Part I: Perspective Projection

Solution (A: Problem 11-2)

In this question we simply have to translate from the world frame to the camera frame using the following equations:

\[
\begin{align*}
    x^c &= y^w \\
    y^c &= z^w - 100 \\
    z^c &= x^w
\end{align*}
\]

Once we have the coordinates in the camera frame then we can proceed using Equation (1) as in Problem 11-1.

\[
\begin{array}{lll}
(1) & (25, 25, 50)^w & (10, -20) \\
(2) & (-25, -25, 50)^w & invisible & (10, -20) \\
(3) & (20, 5, -50)^w & (5, -150, 20)^c & (2.5, -75) \\
(4) & (15, 10, 25)^w & (10, -75, 15)^c & \left(\frac{20}{3}, -50\right) \\
(5) & (0, 0, 50)^w & (0, -50, 0)^c & invisible \\
(6) & (0, 0, 100)^w & (0, 0, 0)^c & invisible
\end{array}
\]

* Note that as in Problem 11-1, points (2), (5), and (6) will be behind the camera so no light will pass through the pinhole. Thus, they will not be visible on the image plane of a physical camera.
Solution (B: Problem 11-3)

From the transformation, we have

\[(x_2, y_2, z_2) = (x_1 - B, y_1, z_1)\]

We know \(u_1, y_1, \lambda_1\) and \(u_2, v_2, \lambda_2\). We want to find depth \(z = z_1 = z_2\).

\[u_1 = \frac{x_1 c_1}{z_1} \lambda_1, \quad u_2 = \frac{x_2 c_2}{z_2} \lambda_2 = \frac{x_1 c_1 - B}{z_1} \lambda_2.\]

\[x_1 = u_1 \left( \frac{z_1}{\lambda_1} \right), \quad x_1 - B = u_2 \left( \frac{z_1}{\lambda_2} \right)\]

Setting equal:

\[x_1 = u_1 \left( \frac{z_1}{\lambda_1} \right) = B + u_2 \left( \frac{z_1}{\lambda_2} \right)\]

\[B = z_1 \left[ \frac{u_1}{\lambda_1} - \frac{u_1}{\lambda_2} \right] \]

\[\Rightarrow z = \frac{\frac{u_1}{\lambda_1} - \frac{u_2}{\lambda_2}}{B}\]
Solution  (C: Problem 11-4)

This problem is far more easily solved geometrically than any other method. Mathematically or algebraically attempting to solve this problem will lead to a set of equations which do not appear linear. However, geometrically solving this problem is much more simple.

Recall that a plane in 3D space is defined by 3 non-collinear points. Thus let Π be a plane defined by any two points on the 3D line in space and the center of the projection. The intersection of the image plane and plane Π is a straight line as the intersection of any two planes is a line.

Please note that there are a few degenerate cases:

1. If the 3D line in space is parallel to the image plane and lies such that plane Π formed by the 3D line and the center of projection is parallel to the image plane, then Π will not intersect the image plane and there will be no projection of the line on the image plane.

2. If the 3D line passes through the center of projection, there are infinite possible planes Π. In this case, the 3D line appears as a single point in the image.
Part II: Vanishing Points

Solution (A: Problem 11-5)

We are given two lines that are parallel in the camera frame.

\[
L_1 : \begin{bmatrix} x_1^c \\ y_1^c \\ z_1^c \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \gamma \mathbf{U}
\]

\[
L_2 : \begin{bmatrix} x_2^c \\ y_2^c \\ z_2^c \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} + \gamma \mathbf{U}
\]

where \( \mathbf{U} = [U_x U_y U_z]^T \) is a unit vector.

From our equations relating points in the camera frame to coordinate in the image frame we have

\[
u = \frac{x^c}{z^c} \lambda, \quad v = \frac{y^c}{z^c} \lambda.
\]

Substitute our lines:

\[
u_i = \frac{(x_i + \gamma U_x)}{(z_i + \gamma U_z)} \lambda
\]

Now take the limits as \( \gamma \to \infty \).

\[
u_\infty = \lim_{\gamma \to \infty} \frac{(x_1 + \gamma U_x)}{(z_1 + \gamma U_z)} \lambda
\]

\[= \frac{U_x}{U_z} \lambda
\]

Similarly, we can find

\[
v_\infty = \frac{U_y}{U_z} \lambda
\]

The coordinate \((u_\infty, v_\infty)\) is the vanishing point.

Note:
1. \((u_\infty, v_\infty)\) does not depend on \(x_i, y_i,\) or \(z_i\). This implies that any line with unit direction \(\mathbf{U}\) will pass through \((u_\infty, v_\infty)\).

2. \((u_\infty, v_\infty)\) does not exist when \(U_z = 0\). When \(U_z = 0\) the plane containing the 3D lines is parallel to the image plane. In this case, the two parallel lines remain parallel in the image; they never intersect.

**Solution (B: Problem 11-6)**

All horizontal lines have \(U_y = 0\). We know from problem 11-5 that

\[ v_\infty = \lambda \frac{U_y}{U_z} = 0 \]

Thus, all horizontal lines must have \(v_\infty = 0\).

**Solution (C: Problem 11-7)**

From problem 11-5 we have

\[ u_\infty = \lambda \frac{U_x}{U_z} \quad v_\infty = \lambda \frac{U_y}{U_z} \]

Since \(\mathbf{U}\) is a unit vector, we also know

\[ U_x^2 + U_y^2 + U_z^2 = 1. \]

Now substitute

\[ u_\infty^2 + v_\infty^2 + \lambda^2 = \frac{\lambda^2}{U_z^2} (U_x^2 + U_y^2) + \lambda^2 = \frac{\lambda^2}{U_z^2} (U_x^2 + U_y^2 + U_z^2) = \frac{\lambda^2}{U_z^2}. \]

Rearranging terms yields

\[ U_z = \frac{\lambda}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}} \]
Substituting this formula for $U_z$ back into the expressions for $u_\infty$ and $v_\infty$, we get

$$U_x = \frac{u_\infty}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}}$$

$$U_x = \frac{v_\infty}{\sqrt{u_\infty^2 + v_\infty^2 + \lambda^2}}$$

**Solution** (D: Problem 11-8)

Let two parallel lines with unit vector $U_i$ define plane $P_i$.

$$U_i = \begin{bmatrix} U_{xi} \\ U_{yi} \\ U_{zi} \end{bmatrix}$$

Consider three such pairs of parallel lines with planes $P_1, P_2, P_3$ all parallel. Since the planes are parallel, they share a common normal, $N$.

$$N = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

From our formulas for vanishing points, we know

$$\begin{cases} u_\infty \chi &= \lambda \frac{U_{zi}}{U_{xi}} \\ v_\infty \chi &= \lambda \frac{U_{yi}}{U_{zi}} \end{cases} \Rightarrow \begin{cases} U_{xi} &= \frac{u_\infty U_{zi}}{\lambda} \\ U_{yi} &= \frac{v_\infty U_{zi}}{\lambda} \end{cases}.$$ 

Since $N$ is normal to any of the lines, we know $U_i \cdot N = 0$ for all $i$.

$$U_i \cdot N = 0 \Rightarrow U_{xi}N_x + U_{yi}N_y + U_{zi}N_z = 0$$

$$\frac{u_\infty U_{zi}}{\lambda}N_x + \frac{v_\infty U_{zi}}{\lambda}N_y + U_{zi}N_z = 0$$
When \( \overline{U}_{zi} \neq 0 \), we have
\[
\frac{Nx}{\lambda}u_{\infty}u_{\infty} + \frac{Ny}{\lambda}v_{\infty}v_{\infty} + N_z = 0
\]
which is the equation of a 2D line of the form \( au_{\infty} + bv_{\infty} + c = 0 \) (in the image plane). Therefore, all vanishing points \( (u_{\infty,i}, v_{\infty,i}) \) lie along this line.

Remark: When \( \overline{U}_{zi} = 0 \), the plane is parallel to the image plane and the parallel lines do not converge in the image.

Solution (E: Problem 11-9)

1. To show that \( \angle V_a CV_b = \frac{\pi}{2} \), it is sufficient to show \( C\overline{V}_a \cdot C\overline{V}_b = 0 \). Since vectors \( a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), \) and \( c = (c_1, c_2, c_3) \) define the edges of the cube, we know they are perpendicular, thus \( a \cdot b = b \cdot c = a \cdot c = 0 \). From our formulas for vanishing points, we have that
\[
C\overline{V}_a = \lambda \left( \frac{a_1}{a_3}, \frac{a_2}{a_3}, 1 \right) \quad \text{and} \quad C\overline{V}_b = \lambda \left( \frac{b_1}{b_3}, \frac{b_2}{b_3}, 1 \right).
\]

\[
C\overline{V}_a \cdot C\overline{V}_b = \lambda^2 \left( \frac{a_1 b_1}{a_3 b_3} + \frac{a_2 b_2}{a_3 b_3} + 1 \right)
= \frac{\lambda^2}{a_3 b_3} (a_1 b_2 + a_2 b_2 + a_3 b_3)
= \frac{\lambda^2}{a_3 b_3} (a \cdot b)
= 0.
\]

Therefore, \( C\overline{V}_a \perp C\overline{V}_b \) which implies \( \angle V_a CV_b = \frac{\pi}{2} \). Similar proofs hold for the other two angles in the problem statement.

2. To show \( V_b \overline{V}_c \) is perpendicular to the plane, it is sufficient to show that two distinct vectors in the plane are perpendicular to \( V_a \overline{V}_b \). That is, the dot products to two distinct vectors in the plane with \( V_b \overline{V}_c \) are both zero. One such vector is the altitude \( h_a \), which is perpendicular to \( V_b \overline{V}_c \) by definition.

Another such vector is \( C\overline{V}_a \). We have
\[
V_b \overline{V}_c = (u_c, v_c, \lambda) - (u_b, v_b, \lambda) = \lambda \left( \frac{c_1}{c_3} - \frac{b_1}{b_3}, \frac{c_2}{c_3} - \frac{b_2}{b_3}, 0 \right)
\]
\[
C\overline{V}_a = (u_a, v_a, \lambda) - (0, 0, 0) = \lambda \left( \frac{a_1}{a_3}, \frac{a_2}{a_3}, 1 \right)
\]
so

\[ V_b \vec{V}_c \cdot C\vec{V}_a = \lambda^2 \left( \left( \frac{a_1 c_1}{a_3 c_3} - \frac{a_1 b_1}{a_3 b_3} \right) + \left( \frac{a_2 c_2}{a_3 c_3} - \frac{a_2 b_2}{a_3 b_3} \right) + 0 \right) \]

\[ = \lambda^2 \left( \frac{a_1 c_1}{a_3 c_3} + \frac{a_2 c_2}{a_3 c_3} + 1 \right) - \left( \frac{a_1 b_1}{a_3 b_3} + \frac{a_2 b_2}{a_3 b_3} + 1 \right) \]

\[ = \lambda^2 \left( \frac{1}{a_3 c_3} (a_1 c_1 + a_2 c_2 + a_3 c_3) - \frac{1}{a_3 b_3} (a_1 b_1 + a_2 b_2 + a_3 b_3) \right) \]

\[ = \lambda^2 (a \cdot c - a \cdot b) = \lambda^2 (0 - 0) = 0 \]

Therefore, \( C\vec{V}_a \), another vector in the plane, is also perpendicular to \( V_b \vec{V}_c \). We conclude that \( V_b \vec{V}_c \) is perpendicular to the plane, and hence is the vector normal to the plane.

3. From the previous part, we know that \( V_b \vec{V}_c \) is normal to plane \( P_a \). The vector \( \vec{N} = (0, 0, 1) \) is normal to the image plane. To show that plane \( P_a \) is perpendicular to the image plane, it is sufficient to show that the respective normal vectors are perpendicular.

\[ V_b \vec{V}_c \cdot \vec{N} = (u_c - u_b)0 + (v_c - v_b)0 + (0)1 = 0 \]

Therefore, the image plane is perpendicular to plane \( P_a \).

4. The orthocenter of the triangle \( V_aV_bV_c \) is \((0, 0, \lambda)\). It lies on the camera’s z axis.

Part III: Optimal Threshold

Solution (A:)

The probability of error is given by

\[ P_{err}(t) = \int_{-\infty}^{\infty} P_0 f_0(z)dz + \int_{-\infty}^{t} P_1 f_1(z)dz \]

\[ = -\int_{-\infty}^{t} P_0 f_0(z)dz + \int_{-\infty}^{t} P_1 f_1(z)dz \]

\[ = -P_0 (F_0(t) - F_0(\infty)) + P_1 (F_1(t) - F_1(-\infty)) \],
and its gradient with respect to the threshold parameter \( t \) is given by
\[
\partial_t P_{\text{err}} = -P_0 \partial_t F_0(t) + P_1 \partial_t F_1(t) \\
= -P_0 f_0(t) + P_1 f_1(t) \\
= -\frac{P_0}{\sqrt{2\pi \sigma_0}} e^{-\frac{(t-\mu_0)^2}{2\sigma_0^2}} + \frac{P_1}{\sqrt{2\pi \sigma_1}} e^{-\frac{(t-\mu_1)^2}{2\sigma_1^2}}.
\]
At the optimal value of \( t \) the error gradient will equal zero. Solving \( \partial_t P_{\text{err}}(t^*) = 0 \) for \( t^* \) requires the algebraic solution to
\[
\gamma_0 e^{a_0 t^2 + b_0 t + c_0} = \gamma_1 e^{a_1 t^2 + b_1 t + c_1}, \tag{1}
\]
where
\[
\gamma_0 := +\frac{P_0}{\sqrt{2\pi \sigma_0}}, \quad \gamma_1 := +\frac{P_1}{\sqrt{2\pi \sigma_1}},
\]
and
\[
a_k := -\frac{1}{2\sigma_k}, \quad b_k := +\frac{\mu_k}{\sigma_k^2}, \quad c_k := +\frac{\mu_k^2}{2\sigma_k^2}.
\]
To solve equation (1), take the log on both sides,
\[
\log \left( \gamma_0 e^{a_0 t^2 + b_0 t + c_0} \right) = \log \left( \gamma_1 e^{a_1 t^2 + b_1 t + c_1} \right),
\]
\[
\log (\gamma_0) + \log \left( e^{a_0 t^2 + b_0 t + c_0} \right) = \log (\gamma_1) + \log \left( e^{a_1 t^2 + b_1 t + c_1} \right),
\]
\[
\log (\gamma_0) + a_0 t^2 + b_0 t + c_0 = \log (\gamma_1) + a_1 t^2 + b_1 t + c_1.
\]
Solving for 0 gives
\[
0 = \log (\gamma_1) - \log (\gamma_0) + (a_1 - a_0) t^2 + (b_1 - b_0) t + (c_1 - c_0),
\]
\[
= \log \left( \frac{\gamma_1}{\gamma_0} \right) + (a_1 - a_0) t^2 + (b_1 - b_0) t + (c_1 - c_0),
\]
\[
= (a_1 - a_0) t^2 + (b_1 - b_0) t + \left( \log \left( \frac{\gamma_1}{\gamma_0} \right) + c_1 - c_0 \right),
\]
\[
= a' t^2 + b' t + c'.
\]
The optimal threshold is then given by the quadratic equation
\[
t^* = -\frac{b' \pm \sqrt{(b')^2 - 4a'c'}}{2a'}
\]
\[
= \frac{\mu_1 \sigma_0^2 + \mu_0 \sigma_1^2 \pm |\sigma_0 \sigma_1| \sqrt{(\mu_0 - \mu_1)^2 + 2(\sigma_1^2 - \sigma_0^2) \log \left( \frac{P_0 \sigma_1}{P_1 \sigma_0} \right)}}{\sigma_0^2 - \sigma_1^2}.
\]
When $\sigma_1^2 \to \sigma_0^2 \to \sigma$, the $a'$ term goes to zero leaving
\[ b't^* + c' = 0, \]
which gives
\[ t^* = \frac{-c'/b'}{\log \left( \frac{\gamma_1}{\gamma_0} \right) + c_1 - c_0} \]
\[ = \frac{\mu_0 + \mu_1}{2} + \frac{\sigma^2}{\mu_1 - \mu_0} \log \left( \frac{P_0}{P_1} \right). \]