ECE 550, Spring 2016 **PROBLEM SET 2** **Due Thursday, March 31**

Reading: *Principles of Robot Motion* Chapters 4 and 7.

Potential field methods for path planning, Sampling-based planners

Problems:

1. Is the potential field planner shown in Algorithm 4 of [Choset] provably correct? If so, sketch the proof. If not, explain why not.

Soln: This planner is not provably correct. Note that the planner never performs an explicit collision detection. Instead, it relies on the repulsive potential field to prevent collisions. This is a heuristic approach: if the repulsive field is large enough near obstacle boundaries, then collisions should be avoided. But, there is no guarantee for this. For example, if the step size is large, it is possible that a single step of the planner could intersect an obstacle, and this would not even by noticed by the planner.

2. Is the potential field planner shown in Algorithm 4 of [Choset] provably complete? If so, sketch the proof. If not, explain why not.

Soln: The planner relies on gradient descent, and the cost function (i.e., the potential function) is not convex. Therefore, it is possible that the planner will converge to a local mininum in the potential field, failing to find a collision-free path even if one exists.

3. Verify Equation 4.1 of [Choset].

Soln: Applying the chain rule, we have

$$\nabla U_{\text{att}}(q) = \nabla \frac{1}{2}\zeta d^2(q, q_{\text{goal}}) = \zeta d(q, q_{\text{goal}}) \nabla d(q, q_{\text{goal}}) \tag{1}$$

For the term $\nabla d(q, q_{\text{goal}})$ we have

$$\begin{aligned} \nabla d(q, q_{\text{goal}}) &= \nabla \left[\sum_i (q^i - q^i_{\text{goal}})^2 \right]^{\frac{1}{2}} && (2) \\ &= \frac{1}{2} \left[\sum_i (q^i - q^i_{\text{goal}})^2 \right]^{-\frac{1}{2}} \nabla \left[\sum_i (q^i - q^i_{\text{goal}})^2 \right] && (3) \end{aligned}$$

Now, note that

$$\frac{\partial}{\partial q^j} \sum_i (q^i - q^i_{\text{goal}})^2 = 2(q^j - q^j_{\text{goal}})$$

and thus

$$\nabla \left[\sum_i (q^i - q^i_{\text{goal}})^2 \right] = 2(q - q_{\text{goal}})$$

Substituting this into (3) we obtain

$$\nabla d(q, q_{\text{goal}}) = \frac{1}{2} \left[\sum_i (q^i - q^i_{\text{goal}})^2 \right]^{-\frac{1}{2}} 2(q - q_{\text{goal}}) = \frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})}$$

and substituting this into (1) we have

$$\nabla U_{\text{att}}(q) = \zeta d(q, q_{\text{goal}}) \frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})} = \zeta (q - q_{\text{goal}})$$

4. Verify Equation 4.5 of [Choset].
Soln: Applying the chain rule multiple times, we obtain the following

$$\begin{aligned}
\nabla \frac{1}{2}\eta\left(\frac{1}{D(q)}-\frac{1}{Q^*}\right)^2 &= \eta\left(\frac{1}{D(q)}-\frac{1}{Q^*}\right)\nabla\left(\frac{1}{D(q)}-\frac{1}{Q^*}\right) && (4)\\
&= \eta\left(\frac{1}{D(q)}-\frac{1}{Q^*}\right)\nabla\frac{1}{D(q)} && (5)\\
&= -\eta\left(\frac{1}{D(q)}-\frac{1}{Q^*}\right)\frac{1}{D^2(q)}\nabla D(q) && (6)
\end{aligned}$$

5. Consider the family of navigation functions given by

$$\phi_k(q) = \frac{\gamma_k(q)}{\beta(q)}$$

in which γ_k is given by Equation 4.9 [Choset], and $\beta(q)$ is given by Equation 4.8 [Choset]. Show mathematically that for large enough k, ϕ_k has a single minimum at $q = q^*$ (i.e., do not merely reproduce the qualitative argument given in [Choset]). Hints: (i) Remember that the domain of ϕ_k is bounded (by the circle of radius r_0). (ii) It is not necessary to explicitly computer $\nabla\beta$.
Soln: By applying the quotient rule, we obtain,

$$\nabla\phi_k(q) = \frac{1}{\beta^2(q)}\left(\beta(q)\nabla\gamma_k(q) - \gamma_k(q)\nabla\beta(q)\right) \tag{7}$$

Computing the gradient of $\gamma_k(q)$, we have

$$\begin{aligned}
\nabla\gamma_k(q) &= \nabla d^{2k}(q, q_{\text{goal}}) && (8)\\
&= \nabla\left(d^2(q, q_{\text{goal}})\right)^k && (9)\\
&= k\left(d^2(q, q_{\text{goal}})\right)^{k-1}\nabla d^2(q, q_{\text{goal}}) && (10)\\
&= 2k\left(d^2(q, q_{\text{goal}})\right)^{k-1}(q - q_{\text{goal}}) && (11)\\
&= 2k\left(d^{2(k-1)}(q, q_{\text{goal}})\right)(q - q_{\text{goal}}) && (12)\\
&= 2k\left(d^{2k-1}(q, q_{\text{goal}})\right)\frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})} && (13)
\end{aligned}$$

Note that (11) follows by applying the result from problem 3 above. Now, substitute this and the definition of γ_k into (7)

$$\begin{aligned}
\nabla\phi_k(q) &= \frac{1}{\beta^2(q)}\left(\beta(q)\nabla\gamma_k(q) - \gamma_k(q)\nabla\beta(q)\right) && (14)\\
&= \frac{1}{\beta^2(q)}\left(\beta(q)2k\left(d^{2k-1}(q, q_{\text{goal}})\right)\frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})} - d^{2k}(q, q_{\text{goal}})\nabla\beta(q)\right) && (15)\\
&= \frac{d^{2k-1}(q, q_{\text{goal}})}{\beta^2(q)}\left(2k\beta(q)\frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})} - d(q, q_{\text{goal}})\nabla\beta(q)\right) && (16)
\end{aligned}$$

A necessary condition for q to be a minimum is that $\nabla\phi_k(q) = 0$. For $q \neq q_{\text{goal}}$, this can only occur if

$$2k\beta(q)\frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})} = d(q, q_{\text{goal}})\nabla\beta(q)$$

But, since the workspace is bounded, β is bounded, as is $\nabla\beta_i(q) = 2(q-q_i)$. Thus, by choosing k sufficiently large, we can ensure that $\nabla\phi_k(q) \neq 0$ for any $q \neq q_{\text{goal}}$. Furthermore, note that the term

$$\frac{(q - q_{\text{goal}})}{d(q, q_{\text{goal}})}$$

is a unit vector directed away the goal. Thus, by choosing k sufficiently large, the negative gradient $-\nabla\phi_k(q)$ approximately points toward the goal.

6. Consider a two-link planar arm with link links a_1 and a_2. Let $q_{\text{init}} = (0,0)$, and $q_{\text{goal}} = (\pi/2, \pi/2)$. Assume that the workspace control points are located at the origins of D-H frames 1 and 2 (i.e., at the end of links 1 and 2), and that the workspace potential is the parabolic well. Compute the configuration space force vector for $q = (0,0)$.

Soln: Denote the origins of DH frames 1 and 2 by o_1 and o2, respectively.

For the two control points, the Jacobians are given by:

$$J_{o_1}(q) = \begin{bmatrix} -a_1 \sin\theta_1 & 0 \\ a_1 \cos\theta_1 & 0 \end{bmatrix}, \quad J_{o_2}(q) = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1+\theta_2) & -a_2 \sin(\theta_1+\theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1+\theta_2) & a_2 \cos(\theta_1+\theta_2) \end{bmatrix} \tag{17}$$

And at $q = (0,0)$ we have

$$J_{o_1}(q) = \begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix}, \quad J_{o_2}(q) = \begin{bmatrix} 0 & 0 \\ a_1 + a_2 & a_2 \end{bmatrix} \tag{18}$$

To compute the attractive forces in the workspace, we need the coordinates of o_1 and o_2 at $q_{\text{init}} = (0,0)$, and $q_{\text{goal}} = (\pi/2, \pi/2)$. Substituting into the forward kinematic equations for the 2-link planar arm, we obtain

$$\begin{aligned} o_1(q_{\text{init}}) &= (a_1, 0) \\ o_2(q_{\text{init}}) &= (a_1 + a_2, 0) \\ o_1(q_{\text{goal}}) &= (0, a_1) \\ o_2(q_{\text{goal}}) &= (-a_2, a_1) \end{aligned}$$

The workspace attractive forces are then given by

$$\begin{aligned} F_{o_1}(q_{\text{init}}) &= -\nabla U_{o_1}(q_{\text{init}}) = -(o_1(q_{\text{init}}) - o_1(q_{\text{goal}})) = (-a_1, a_1) \\ F_{o_2}(q_{\text{init}}) &= -\nabla U_{o_1}(q_{\text{init}}) = -(o_2(q_{\text{init}}) - o_2(q_{\text{goal}})) = (-a_1 - 2a_2, a_1) \end{aligned}$$

And finally, we compute the configuration space force by

$$\begin{aligned} \mathcal{F}(q_{\text{init}}) &= J_{o_1}^T(q_{\text{init}}) \begin{bmatrix} -a_1 \\ a_1 \end{bmatrix} + J_{o_2}^T(q_{\text{init}}) \begin{bmatrix} -a_1 - 2a_2 \\ a_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -a_1 \\ a_1 \end{bmatrix} + \begin{bmatrix} 0 & a_1 + a_2 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} -a_1 - 2a_2 \\ a_1 \end{bmatrix} \\ &= \begin{bmatrix} 2a_1^2 + a_1 a_2 \\ a_1 a_2 \end{bmatrix} \end{aligned}$$

7. As we discussed in class, RPP attempts to solve the local minimum problem that faces potential fields planners. For PRM, we can prove probabilistic completeness by (a) building a "ρ-tube" around a feasible path, and showing that PRM fails to a path in this tube with probability that exponentially decreases to zero with the number of samples.

Is it possible to construct a similar proof to show that RPP is probabilistically complete for the simpler problem of escaping the attractive basin of a particular local minimum in the artificial potential field? In particular, suppose there exists a path γ that escapes this attractive basin by following the gradient of the potential field, and further, that any path within distance ρ of the path γ will also escape the attractive basin by following the gradient of the potential field. Can we us a PRM-style approach to prove that a random walk will find a path in this ρ-tube? If so, sketch the proof. If not, explain why not.

Soln: In the PRM proof, we tiled the path γ with balls of radius $\rho/2$, and evaluated the probability that the sampling scheme would place at least one sample in each of these balls. To compute a bound on this probability, we required the probability that the algorithm would fail to place a sample in a particular ball, say B_i. This computation relied on the statistical independence of the samples. In particular, we were able to assume that the probability that no sample falls in ball B_i is given by p^N, where p is the probability that a single sample does not fall in B_i, and N is the number of samples. In the case of RPP, samples are not independent, since the placement of the kth sample depends on the placement of the $(k-1)^{th}$ sample. Thus, the PRM style proof cannot be applied directly; without the independence assumption, it isn't clear how one might construct the probability that no sample falls in a given ball B_i.

8. Consider a rigid object that moves freely in a 3-dimensional unit cube in Euclidean space, and whose orientation is represented by a unit quaternion. The configuration space of this object is represented by $\mathcal{Q} = Q \times [0,1] \times [0,1] \times [0,1]$, in which Q denotes the set of unit quaternions. Describe a procedure for generating random samples from $\mathcal{Q}$.

Soln: Generating samples on $\mathcal{Q}$ requires (a) generating samples from the space of unit quaternions Q, and (b) generating samples from the unit cube, $[0,1] \times [0,1] \times [0,1]$.

(a) Since this problem does not specify any target distribution for the samples, there are many answers, some of which are nearly trivial. For example, since a every 4-vector of unit magnitude is a unit quaternion, one could simply generate $u_1, u_2, u_3, u_4 \sim U(-1,1)$ and then normalize[1]. This, and many similar approaches, will generate random samples on Q, but the distribution of these samples will not be uniform.

If we want uniform samples, a nice procedure is described in Section 5.2.2 of [LaValle 2006]. The method is simple to implement. First, generate samples $u_1, u_2, u_3 \sim U(0,1)$. Then, define the sample quaternion as

$$q = (\sqrt{1-u_1}\sin 2\pi u_2, \sqrt{1-u_1}\cos 2\pi u_2, \sqrt{u_1}\sin 2\pi u_3, \sqrt{u_1}\cos 2\pi u_3)$$

(b) To sample the unit cube, merely choose $x_1, x_2, x_3 \sim U(0,1)$. This part is pretty easy.

9. Consider a robot whose dynamics are specified by the differential equation $\dot{x} = f(x,u)$, in which $x \in X$ denotes the state of the robot, and $u \in U$ is the input. Can the postprocessing method described in Section 7.1.2 (Fig. 7.7) of [Choset] be used to smooth a path for this system? Justify your answer.

Soln: In principle, this approach can be used to shorten and smooth a path, but in practice this depends on the local planner. Consider a path that is specified as a sequence of samples $(x_0, x_1, \ldots x_{n-1}, x_n)$, such that $x_0 = x_{\text{init}}$ and $x_n = x_{\text{goal}}$. Now, suppose we wish to shorten the path by directly connecting x_i to x_j for some $i < j$. To do this, the local planner must solve a two-point boundary value problem, such that $x_j = x_i + \int f(x,u)dt$, which requires determining $u(t)$, the required input to drive the system from x_i to x_j. This problem is, in general, much more difficult than merely determining whether the direct path from x_i to x_j is collision-free, which is the problem addressed in Section 7.1.2 of [Choset]. Therefore, while this

[1]Note: $U(-1,1)$ denotes the uniform density on the interval $(-1,1)$.

idea is possible in principle, in practice it may require heavy computation (e.g., numerical optimization methods).

10. Consider a polygon $\mathcal{A}$ with configuration space $\mathcal{Q} = SE(2)$. Define the function $\rho : SE(2) \times SE(2) \to \Re$ given by

$$\rho(q_1, q_2) = \max_{a \in \mathcal{A}} ||a(q_1) - a(q_2)||$$

in which $a(q)$ denotes the position in the plane of $a \in \mathcal{A}$ when $\mathcal{A}$ is at configuration q. Show that ρ is a metric.

Soln: To show that ρ is a metric, we must show that

(a) ρ is positive definite, i.e., $\rho(q_1, q_2) \geq 0$, with equality only when $q_1 = q_2$;

(b) ρ is symmetric, i.e., $\rho(q_1, q_2) = \rho(q_2, q_1)$; and

(c) ρ satisfies the triangle inequality, i.e., $\rho(q_1, q_3) \leq \rho(q_1, q_2) + \rho(q_2, q_3)$.

Each of these properties holds:

(a) If $q_1 = q_2$, then for every point $a \in \mathcal{A}$, $a(q_1) = a(q_2)$, and thus, $\rho(q_1, q_2) = 0$.
If $q_1 \neq q_2$, then $\max_{a \in \mathcal{A}} ||a(q_1) - a(q_2)|| > 0$, since not all points in $\mathcal{A}$ remain stationary, and thus there are a such that $a(q_1) \neq a(q_2)$.

(b) Since $||a(q_1) - a(q_2)|| = ||a(q_2) - a(q_1)||$, ρ is symmetric.

(c) We may show that ρ satisfies the triangle inequality as follows:

$$\begin{aligned}
\rho(q_1, q_2) + \rho(q_2, q_3) &= \max_{a \in \mathcal{A}} ||a(q_1) - a(q_2)|| + \max_{a' \in \mathcal{A}} ||a'(q_2) - a'(q_3)|| && (19)\\
&\geq \max_{a \in \mathcal{A}} \left\{ ||a(q_1) - a(q_2)|| + ||a(q_2) - a(q_3)|| \right\} && (20)\\
&\geq \max_{a \in \mathcal{A}} ||a(q_1) - a(q_3)|| && (21)\\
&= \rho(q_1, q_3) && (22)
\end{aligned}$$

In the derivation above, (20) follows because $\max_{a' \in \mathcal{A}} ||a'(q_2) - a'(q_3)|| \geq ||a(q_2) - a(q_3)||$ for any $a \in \mathcal{A}$. Inequality (21) follows because the Euclidean norm is itself a metric, and thus satisfies the triangle inequality.