ECE 470: Homework 1
Due Tuesday, September 8
in class @12:30pm

Seth Hutchinson

Luke A. Wendt
1

A free vector is defined by the subtraction of two points in a given coordinate system, e.g., \( v = p_1 - p_2 \).

1 (a)

Using the fact that \( v_1 \cdot v_2 = v_1^T v_2 \), show that the dot product of two free vectors does not depend on the choice of frames in which their coordinates are defined.

1 (b)

Show that the Euclidean distance between two points does not depend on the choice of frames in which their coordinates are defined.

2

Let the \( 3 \times 3 \) matrix \( M \) satisfy \( M^T M = I \), and let the column vectors of \( M \) be \( m_c \) with \( M = \begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} \).

2 (a)

Show that the column vectors of \( M \) are of unit length and mutually perpendicular, i.e., \( m_c^T m_{c'} = \begin{cases} 1 & \text{if } c = c' \\ 0 & \text{if } c \neq c' \end{cases} \)

2 (b)

The determinant has the following properties:

1. For two square matrices \( M \) and \( M' \), \( \det(MM') = \det(M) \det(M') \).
2. For all square matrices, \( \det(M) = \det(M^T) \).

Show that \( \det(M) = \pm 1 \).

2 (c)

If the column vectors of \( M \) are right handed, i.e., \( m_1 \times m_2 = m_3 \), show that \( \det(M) = +1 \).

3

Consider the following sequence of rotations and write the matrix product that will give the resulting rotation matrix (do not compute the matrix products).

1. rotate by \( \theta_3 \) about the world z-axis
2. rotate by \( \theta_2 \) about the world y-axis
3. rotate by \( \theta_1 \) about the world x-axis
4. rotate by \( \theta_4 \) about the current x-axis
5. rotate by \( \theta_5 \) about the current y-axis
6. rotate by \( \theta_6 \) about the current z-axis
4

Suppose four coordinate frames are given, and suppose \( R^2_1 = R_{x,\theta_1} \), \( R^3_1 = R_{y,\theta_2} \), and \( R^4_3 = R_{z,\theta_3} \).

4 (a)

Give an algebraic expression for \( R^3_2 \) and compute the matrix product by hand.

4 (b)

Give an algebraic expression for \( R^1_4 \) and compute the matrix product by hand.

4 (c)

Give an algebraic expression for \( R^2_4 \) and compute the matrix product by hand.

5

Derive equations for the roll, pitch, and yaw angles corresponding to the \( r_{ij} \) elements of the rotation matrix \( R \) given by equation (2.38).

6

For \( R \in SO(3) \) and \( d \in \mathbb{R}^3 \), verify \( H^{-1} = \left[ \begin{array}{cc} R & d \\ 0 & 1 \end{array} \right]^{-1} = \left[ \begin{array}{cc} R^T & -R^T d \\ 0 & 1 \end{array} \right] \).

7

Consider the diagram of Figure 1. A robot is set up 1 meter from a table. The table top is 1 meter high and 1 meter square. Frame 1 is fixed to the edge of the table as shown. A cube measuring 20cm on a side is placed in the center of the table with frame 2 established at the center of the cube as shown. A camera is situated directly above the center of the block 2 meters above the table top with frame 3 attached as shown. Find the homogeneous transformations relating each of these frames to the base frame 0. Find the homogeneous transformation relating frame 2 to camera frame 3.

8

In general, homogeneous transformation matrices do not commute. Consider the matrix product

\[
H = R_{z,\theta_1} R_{x,\theta_2} T_{x,d_1} R_{x,\theta_3} T_{y,d_2}
\]

Determine which pairs of the matrices on the right hand side commute. Explain why these pairs commute. Find all permutations of the matrices that yield the same homogeneous transformation matrix \( H \).
Introduction to Quaternions:

Complex numbers can be generalized by defining three independent square roots for \(-1\) that obey the order specific multiplication rules \(ijk = i^2 = j^2 = k^2 = -1\). These rules generate the following products:

\[
\begin{align*}
    i &= +jk = -kj, & j &= +ki = -ik, & k &= +ij = -ji.
\end{align*}
\]

Quaternions are constructed from linear combinations of \(\{i, j, k\}\), e.g.,

\[
Q = q_0 + q_1i + q_2j + q_3k,
\]

and can be expressed more compactly with the 4-tuple vector \(q = [q_0, q_1, q_2, q_3]^T\). The components of the quaternion product

\[
z_0 + z_1i + z_2j + z_3k = (x_0 + x_1i + x_2j + x_3k)(y_0 + y_1i + y_2j + y_3k)
\]

are given by

\[
\begin{bmatrix}
z_0 \\
z_1 \\
z_2 \\
z_3
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
y_0
\end{bmatrix}.
\]

Quaternions arose from the study of a rotation by \(\theta\) about the unit vector \(n = [n_1, n_2, n_3]^T\). This gives the rotation matrix, equation (2.43),

\[
R(\theta, n) =
\begin{bmatrix}
n_1^2(1 - c_\theta) + c_\theta & n_1n_2(1 - c_\theta) - n_3s_\theta & n_1n_3(1 - c_\theta) + n_2s_\theta & 0 \\
n_1n_2(1 - c_\theta) + n_3s_\theta & n_2^2(1 - c_\theta) + c_\theta & n_2n_3(1 - c_\theta) - n_1s_\theta & 0 \\
n_1n_3(1 - c_\theta) - n_2s_\theta & n_2n_3(1 - c_\theta) + n_1s_\theta & n_3^2(1 - c_\theta) + c_\theta & 0
\end{bmatrix}.
\]

By defining 4 new variables

\[
q_0 = \cos \left(\frac{\theta}{2}\right), \quad q_1 = n_1 \sin \left(\frac{\theta}{2}\right), \quad q_2 = n_2 \sin \left(\frac{\theta}{2}\right), \quad q_3 = n_3 \sin \left(\frac{\theta}{2}\right),
\]

the trigonometric functions can be removed and the rotation matrix becomes

\[
R(q) = 2 \begin{bmatrix}
q_1^2 + q_0^2 - \frac{1}{2} & q_1q_2 - q_0q_3 & q_0q_2 + q_1q_3 \\
q_1q_2 + q_0q_3 & q_2^2 + q_0^2 - \frac{1}{2} & q_2q_3 - q_1q_0 \\
q_1q_3 - q_2q_0 & q_2q_3 + q_1q_0 & q_3^2 + q_0^2 - \frac{1}{2}
\end{bmatrix}.
\]

Note \(q^Tq = 1\). The angle of rotation and normal vector can be obtained with

\[
\theta = 2 \arccos(q_0), \quad \text{and} \quad n = \frac{[q_1, q_2, q_3]^T}{\sin \left(\frac{\theta}{2}\right)} = \frac{[q_1, q_2, q_3]^T}{\sqrt{1 - q_0^2}}.
\]

Note that the rotation matrix is uniquely specified with the exception that \(R(q) = R(-q)\). This corresponds to flipping the sign of \(n\) and rotating in the opposite direction. Either representation performs the same transformation. Typically the sign of \(q_0\) is forced to be positive to obtain a unique solution. The inverse matrix is given by \(R^{-1}(q) = R(q)^T = R(q^*)\) where \(q^* = [q_0, -q_1, -q_2, -q_3]^T\) is called the conjugate.
### 9

**Extra Credit:**

With the quaternions \( Q = q_0 + q_1 i + q_2 j + q_3 k \), \( V = v_1 i + v_2 j + v_3 k \), \( Q^* = q_0 - q_1 i - q_2 j - q_3 k \), and the vector \( v = [v_1, v_2, v_3]^T \), verify that the components of the quaternion product \( V' = QVQ^* \) are equal to the components of the matrix product \( v' = R(q)v \). Do not express in terms of \( \theta \) and \( n \).

### 10

**Extra Credit:**

Using \( R(q) \) with \( q = [\cos(\frac{\theta}{2}) \sin(\frac{\theta}{2}) n_1 \sin(\frac{\theta}{2}) n_2 \sin(\frac{\theta}{2}) n_3]^T \) verify equation (2.45) for \( \theta \) and \( n \).
Figure 1: Robot & Camera