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# On the existence of Nash Equilibrium for a two player Pursuit-Evasion Game with Visibility Constraints

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**Abstract:** In this paper, we present a game theoretic analysis of a visibility based pursuit-evasion game in a planar environment containing obstacles. The pursuer and the evader are holonomic having bounded speeds. Both the players have a complete map of the environment. Both the players have omnidirectional vision and have knowledge about each other's current position as long as they are visible to each other. The pursuer wants to maintain visibility of the evader for maximum possible time and the evader wants to escape the pursuer's sight as soon as possible. Under this information structure, we present necessary and sufficient conditions for surveillance and escape. We present strategies for the players that are in *Nash Equilibrium*. The strategies are a function of the *value* of the game. Using these strategies, we construct a value function by integrating the *adjoint equations* backward in time from the termination situations provided by the corners in the environment. From these value functions we recompute the control strategies for the players to obtain optimal trajectories for the players near the termination situation. As far as we know, this is the first work that presents the necessary and sufficient conditions for tracking for a visibility based pursuit-evasion game and presents the equilibrium strategies for the players.

## 1 Introduction

Consider a situation in which a group of mobile pursuers having bounded speeds are trying to keep sight of an unpredictable evader in a cluttered environment. In order to deploy minimum number of pursuers needed to track the evader it would be useful to know the best strategy that can be used by a single pursuer. In this work, we analyze the problem of a mobile pursuer trying to track a mobile evader in an environment containing obstacles. Both the pursuer and the evader are holonomic with bounded speeds and can see each other at the beginning of the game. The players do not have knowledge of each other's future actions. We formulate the problem of tracking as a game in which the goal of the pursuer is to keep the evader in its field-of-view for maximum possible time and the goal of the evader is to escape the pursuer's field-of-view in minimum time by breaking the line of sight around a corner.

An interesting application of this problem is in security and surveillance systems. It may be useful for a security robot to track a malicious evader that is trying

to escape. Also, an “evader” may not be intentionally trying to slip out of view. A pursuer robot may simply be asked to continuously follow and monitor at a distance an evader performing a task not necessarily related to the target tracking game[6]. The pursuer may also be monitoring the evader for quality control, verifying the evader does not perform some undesired behavior, or ensuring that the evader is not in distress. The results are useful as an analysis of when escape is possible. If it is impossible to slip away, it may be desirable for the evader to immediately surrender or undertake a strategy not involving escape. In home care settings, a tracking robot can follow elderly people and alert caregivers of emergencies. Target-tracking techniques in the presence of obstacles have been proposed for the graphic animation of digital actors, in order to select the successive viewpoints under which an actor is to be displayed as it moves in its environment [19].

In the past, we have addressed tracking problems similar to the one in this paper. In [7], we address the problem of a pursuer trying to track an antagonistic evader around a single corner. We partition the visibility region of the pursuer into regions based on the strategies used by the players to achieve their goals. Based on these partitions we propose a sufficient condition of escape for the evader in general environments. In [8], given the initial position of the evader in a general environment, we use the sufficient condition to compute an approximate bound on the initial positions of the pursuer from which it might track the evader. The bound depends on the ratio of the maximum speed of the evader to that of the pursuer. If the initial position of the pursuer lies outside this bound, the evader can escape the pursuer’s sight. Moreover, we provide strategies for the evader to escape irrespective of pursuer’s actions. In this work, we formulate the target-tracking problem as a game in which the pursuer wants to maximize the time for which it can track the evader and the evader wants to minimize it. We compute the strategies for the players that are in *Nash equilibrium*. If a player follows its equilibrium strategy, it is guaranteed of a minimum outcome without any knowledge of the other player’s future actions. Moreover when a pair of strategy for the players is in *Nash equilibrium* then any unilateral deviation of a player from its equilibrium strategy might lead to a lower outcome for it. Consider a situation in which the pursuer can keep the evader in sight for time  $t_f$  when the players follow their equilibrium strategies. If the evader deviates from its equilibrium strategy then the pursuer has a strategy to track it for a time greater than  $t_f$ . On the other hand, if the pursuer deviates from its equilibrium strategy then the evader can escape in time less than  $t_f$ . Hence there is no motivation for either of the players to deviate from their equilibrium strategies due to the lack of knowledge of the other player’s future actions. For a pair of equilibrium strategies for the players either the evader can escape the pursuer’s sight in finite time or the pursuer can track the evader forever. Hence computing them gives us the strategies sufficient for tracking or escape, whichever holds at a given point in the state space. As far as we know, this is the first work that addresses the necessary and sufficient conditions for tracking and provides equilibrium strategies for the players. We use these strategies to integrate the kinematic equations of the system backward in time from the termination situations to obtain the optimal trajectories for the players.

Prior work regarding the problem of tracking is based on discretizing the motion models of the players or the state-space in which the game is being played [17, 5]. These techniques lead to approximate numerical solutions that become computationally inefficient with increasing time horizon of the game. Moreover they assume a prior model of uncertainty for the evader's future actions. Contrary to these works, we use continuous time motion models for the players and provide closed form solutions to the coupled non-linear differential equations that govern our system kinematics. Hence no error is introduced in the solutions due to discretizations of any form. Further, our results are valid for scenarios in which the players have no knowledge about each others future actions.

Some variants of the tracking problem have also been addressed. In [11], the authors take into account the positioning uncertainty of the robot pursuer. Game theory is proposed as a framework to formulate the tracking problem, and an approach is proposed that periodically commands the pursuer to move into a region that has no localization uncertainty in order to re-localize and better track the evader afterward. [10] presents an off-line algorithm that maximizes the evader's minimum time to escape for an evader moving along a known path. Since the entire trajectory of the evader is known beforehand, the problem reduces to a single player optimization problem. In [14] and [4], gradient descent algorithms have been proposed by formulating a local risk function for a pursuer having the local map of the evader. [4] deals with the problem of *stealth target tracking* where a robot equipped with visual sensors tries to track a moving target among obstacles and, at the same time, remain hidden from the target. Obstacles impede both the tracker's motion and visibility, and also provide hiding places for the tracker. A tracking algorithm is proposed that applies a local greedy strategy and uses only local information from the tracker's visual sensors and assumes no prior knowledge of target tracking motion or a global map of the environment. In [23], the problem of target tracking has been analyzed at a fixed distance between the pursuer and evader. Optimal motion strategies are proposed for a pursuer and evader based on critical events.

In this work, we use differential games to analyze a pursuit-evasion problem. The theory of deterministic pursuit-evasion was single-handedly created by R. Isaacs that culminated in his book [15]. A general framework based on the concepts in classical game theory and the notion of tenet of transition was used to analyze pursuit-evasion problems. Problems like the *Lady in the Lake*, *Lion and the Man*, *Homicidal chauffeur* and *Maritime Dogfight Problem* were introduced in this book. A modification to the classical problems involves the consideration of discrete-time versions of these problems and the application of a proper information structure to compute the value of the game [13, 12]. An exhaustive analysis of solved and partly solved zero-sum differential games is provided in [3] and [18]. Most of the classical problems in pursuit-evasion deal with players in obstacle-free space having either constraints on their motion or constraints on their control due to under-actuation. In the recent past, researchers have analyzed pursuit-evasion problems with constraints in the state space. In [20, 21, 22], a pursuit-evasion game is analyzed with the pursuer and the evader constrained to move on a two-dimensional conical surface in a three-dimensional space. Our work belongs to this category of problems. In our problem,

the state constraints arise due to the presence of obstacles that obstruct visibility as well as motion of the players in the workspace and the control constraints arise as a result of the bounded speed of the players. Apart from these problems, researchers have also analyzed pursuit-evasion in  $\mathbb{R}^n$  [16], in non-convex domains of arbitrary dimension [1], in unbounded domains[2] and in graphs [24].

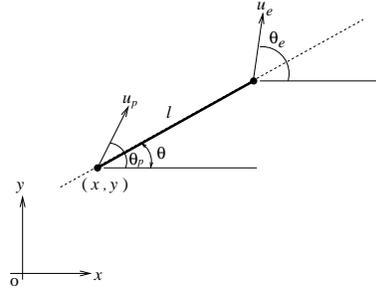
In Section II, we present the formulation of the game. In Section III, we analyze the termination situations presented by the obstacles around any corner in the environment. In Section IV, we present the strategies for the players that are in *Nash equilibrium*. In Section V we present the construction of the optimal trajectories. In Section VI, we present the conclusions and the future work.

## 2 Formulation of the game

We consider a mobile pursuer and an evader moving in a plane with velocities  $u = (u_p, \theta_p)$  and  $v = (u_e, \theta_e)$  respectively.  $u_p$  and  $u_e$  are the speeds of the players that are bounded by  $\bar{v}_p$  and  $\bar{v}_e$  respectively.  $\theta_p$  and  $\theta_e$  are the direction of the velocity vectors. We use  $r$  to denote the ratio of the maximum speed of the evader to that of the pursuer  $r = \frac{\bar{v}_e}{\bar{v}_p}$ . They are point robots with no constraints in their motion except for bounded speeds. The workspace contains obstacles that restrict pursuer and evader motions and may occlude the pursuer's line of sight to the evader. The initial position of the pursuer and the evader is such that they are visible to each other. The visibility region of the pursuer is the set of points from which a line segment from the pursuer to that point does not intersect the obstacle region. Visibility extends uniformly in all directions and is only terminated by workspace obstacles (omnidirectional, unbounded visibility). The pursuer and the evader know each others current position as long as they can see each other. Both the players have a complete map of the environment. In this setting, we consider the following game. The pursuer wants to keep the evader in its visibility region for maximum possible time and the evader wants to break the line of sight to the pursuer as soon as possible. If at any instant, the evader breaks the line of sight to the pursuer, the game terminates. Given the initial position of the pursuer and the evader, we want to know the optimal strategies used by the players to achieve their respective goals. Optimality refers to the strategies used by the players that are in *Nash equilibrium*.

We model the system as a non-rigid bar of variable length representing the line of sight between the pursuer and the evader. The bounded velocities of the pursuer and the evader are modeled as control inputs at opposite ends of the bar. Any occlusion between the pursuer and the evader leads to a situation in which the bar intersects the obstacles. Hence the pursuer's goal is to keep the bar in free space for the maximum possible time and the evader's goal is to force the bar to intersect some obstacle as soon as possible. In this work we assume that the line of sight is not blocked due to grazing contact with the boundary. Hence visibility is retained even if a vertex in the environment is incident on the bar.

Figure 1 shows the configuration of the system along with the state variables and the control inputs.  $(x, y)$  is the position of the end of the bar controlled by the



**Fig. 1.** State variables and Control inputs

pursuer.  $l$  is the length of the bar and  $\theta$  is the angle made by the bar with the horizontal line. The configuration of the system can be expressed as  $(x, y, l, \theta)$  and hence it is  $\mathbb{R}^3 \times S^1$ . In the rest of the paper,  $\mathbf{x}(\in \mathbb{R}^3 \times S^1)$  will be used to represent the state of the bar. The pursuer controls the velocity,  $u_p$ , of one end of the bar and the evader controls the velocity,  $v$ , of the other end of the bar. The differential equation describing the kinematics of the system is given by the following equation.

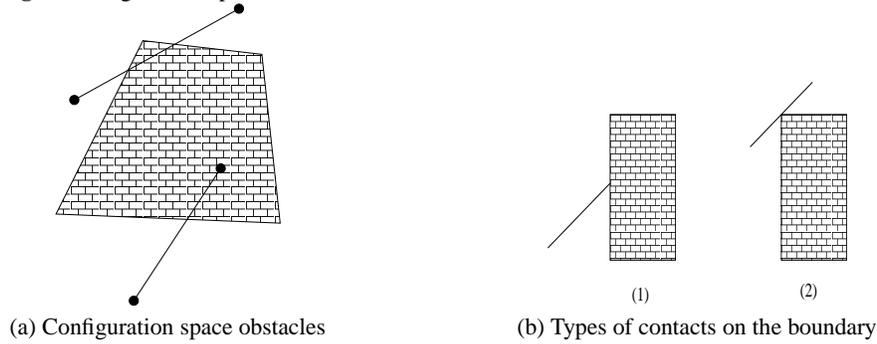
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{l} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} u_p \cos \theta_p \\ u_p \sin \theta_p \\ u_e \cos(\theta_e - \theta) - u_p \cos(\theta_p - \theta) \\ \frac{u_e}{l} \sin(\theta_e - \theta) - \frac{u_p}{l} \sin(\theta_p - \theta) \end{pmatrix}$$

The above equation can also be expressed in the form  $\dot{\mathbf{x}} = f(\mathbf{x}, u, v)$ .

### 3 State constraints and Termination situations

In this section, we present a description of the obstacles in the configuration space. The workspace contains polygonal obstacles in the plane that obstruct the visibility and motion of the players. Since the system is modeled as a bar representing the line of sight between the players, the obstruction of mutual visibility as well as the motion of the players caused due to obstacles in the workspace can be expressed as a state constraint in  $\mathbb{R}^3 \times S^1$ . These state constraints can be expressed as configuration space obstacles. In  $\mathbb{R}^3 \times S^1$ , the configuration space obstacles are the set of all  $(x, y, l, \theta)$  such that the bar has a non-empty intersection with some obstacle in the workspace. Figure 2 shows two such configurations of the bar that lies in configuration space obstacles. In one configuration the obstacle blocks the line-of-sight between the pursuer and the evader. In the other configuration a player is inside the obstacle which is forbidden according to the rules of the game.

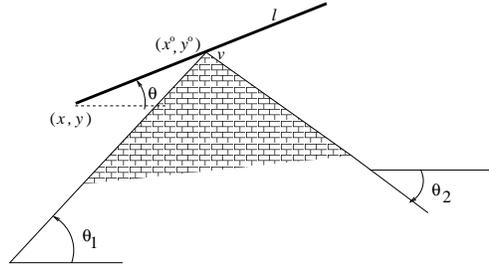
The *game set* is the set of all points in  $\mathbb{R}^3 \times S^1$  that belong to the free space. Hence the boundary of the game set is same as the boundary of the configuration space obstacles. The boundary of the game set consists of two kinds of contact between the bar and the obstacles. Refer to Figure 2(b). The first kind of contact occurs when

**Fig. 2.** Configuration Space Obstacles

at least one end of the bar touches an obstacle in the plane. At no point in time, the state of the game can cross the boundary at such a point as this is equivalent to either of the players penetrating into an obstacle in the workspace. The second kind of contact occurs when a vertex of an obstacle is incident on the bar and these set of points on the boundary of the game set is called the *Target set*. At any point in time, if the current state of the game lies on the target set, then it can cross the boundary according to the rules of the game since in the workspace this is equivalent to breaking the mutual visibility between the players which results in the termination of the game. Since we are interested in situations where the mutual visibility between the players can be broken, we are only interested in the part of the boundary that forms the target set.

In this game, termination occurs only when the evader can break the line of sight to the pursuer around a corner. Every corner in the environment presents an opportunity for the evader to break the line of sight. Hence every corner presents a termination situation for the game.

If the state of the system lies on the target set then a vertex of some obstacle is incident on the bar. The evader cannot guarantee termination at every point on the target set. Figure 3 shows a configuration of the bar in which the system is on the target set. Let  $d_p$  denote the distance of the vertex from  $(x, y)$  which is same as the distance of the pursuer from the vertex. Let  $l$  denote the length of the bar which is same as the distance between the pursuer and the evader. The evader can force termination if and only if the maximum angular velocity of the evader around the corner is greater than the maximum angular velocity achievable by the pursuer around the corner. This can happen if and only if  $\frac{d_p}{l} > \frac{1}{1+r}$ . Hence we can further subdivide the target set depending on whether the evader can guarantee termination at that point. The part of the target set where evader can guarantee termination regardless of the choice of the controls of the pursuer is called the *usable part* (UP). The remaining part of the target set outside the UP is called the *non-usable part* (NUP) and the game will never terminate on the NUP. Given any initial position of the pursuer and the evader, the game will always terminate on the UP.



**Fig. 3.** A configuration of the bar on the target set.

Now we present the equations characterizing the target set around a vertex of an obstacle. Refer to Figure 3. The figure shows a configuration of the bar in which a vertex,  $v$ , lies on the bar. Hence the current state of the bar lies on the target set. We want the equation of the hyperplane that characterizes the target set generated by  $v$ . Let  $(x, y, l, \theta)$  be the configuration of the bar and  $(x^o, y^o)$  be the coordinates of the vertex of the obstacle. Let  $\lambda \in (0, 1)$  be a variable that determines the fraction of the length of the bar between  $(x, y)$  and the corner  $(x^o, y^o)$ . We can write the following equations of constraints for the bar.

$$x^o - x = \lambda l \cos \theta$$

$$y^o - y = \lambda l \sin \theta$$

In the above equation, as  $\lambda$  changes, the point of contact between the bar and the vertex changes. Hence the target set is characterized by the following equation.

$$\Rightarrow F(x, y, l, \theta) = (y^o - y) \cos \theta - (x^o - x) \sin \theta = 0 \tag{1}$$

Since the above equation applies to any  $\lambda \in (\frac{r}{1+r}, 1)$ , Equation (1) also characterizes the usable part of the target set.

Given a vertex, the target set generated by it in the configuration space has the following boundaries.

- The pursuer lies on the corner  $\Rightarrow (x, y) = (x^o, y^o)$ .
- The evader lies on the corner  $\Rightarrow (x + l \cos \theta, y + l \sin \theta) = (x^o, y^o)$ .
- The bar is parallel to either of the edges incident on the vertex :  $\theta = \theta_2$  or  $\theta = \theta_1$ .

Every vertex will generate a target set. The final boundary of the target set generated by a vertex will depend on the position of the other vertices and edges in the environment. But the equation of the target set will be given by (1).

The unit normal to a point  $(x, y, l, \theta)$  on the target set is given by

$$n(x, y, l, \theta) = \nabla F = \frac{1}{\sqrt{1 + (x^o - x)^2 \sec^2 \theta}} [\sin \theta \quad -\cos \theta \quad 0 \quad -(x^o - x) \sec \theta]^T \tag{2}$$

## 4 Optimal Strategies

In this section we present the optimal controls for the players. Before we define the concept of optimality we need to define the payoff for the players in the game. Consider a play that terminates at time  $t_f$ . Since the pursuer wants to increase the time of termination its payoff function can be considered as  $t_f$ . On the other hand since the evader wants to minimize the time of termination its payoff can be considered to be  $-t_f$ . Since the payoff functions of the players add to zero, this is a *zero sum* game. Another way to show that it is a zero sum game is to observe that the pursuer's gain is equal to the evader's loss and vice-versa. The time of termination is a function of the initial state  $\mathbf{x}_0$  and the control history during the play,  $u$  and  $v$ . Let  $\pi$  denote the functional  $\pi : (\mathbf{x}_0, u, v) \rightarrow t_f \in \mathbb{R}$ .  $\pi$  is called the *outcome functional* and is given by the following expression.

$$\pi[\mathbf{x}_0, u, v] = \int_0^{t_f} L[\mathbf{x}(\tau), u(\tau), v(\tau)]d\tau + G[\mathbf{x}(t_f)]$$

In the above expression  $L[\mathbf{x}(\tau), u(\tau), v(\tau)]$  is called the *running cost function* and  $G[\mathbf{x}(t_f)]$  is called the *terminal cost function*. The running cost function is the cost incurred while the game is being played. The terminal cost function is the cost incurred for reaching a particular terminal state on the target set. In this game,  $L[\mathbf{x}(\tau), u(\tau), v(\tau)] = 1$  and  $G[\mathbf{x}(t_f)] = 0$ . The pursuer wants to maximize the outcome functional and the evader wants to minimize it.

For a point  $\mathbf{x}$  in the state space,  $J(\mathbf{x})$  represents the outcome if the players implement their optimal strategy starting at the point  $\mathbf{x}$ . It is the time of termination of the game when the players implement their optimal strategies. It is also called the *value* of the game at  $\mathbf{x}$ . Any unilateral deviation from the optimal strategy by a player can lead to a better payoff for the other player. For example, for a game that starts at a point  $\mathbf{x}$ , if the evader deviates from the optimal strategy then there is a strategy for the pursuer in which its payoff is greater than  $J(\mathbf{x})$  and if the pursuer deviates from the optimal strategy then there is a strategy for evader in which its payoff is greater than  $-J(\mathbf{x})$ . Since this is a *zero sum* game, any strategy that leads to a higher payoff for one player will reduce the payoff for the second player.

$\nabla J = [J_x \ J_y \ J_l \ J_\theta]^T$  denotes the gradient of the value function. The Hamiltonian of any system is given by the following expression.

$$H(\mathbf{x}, \nabla J, u, v) = \nabla J \cdot f(\mathbf{x}, u, v) + L(\mathbf{x}, u, v)$$

Let  $u^* = (u_p^*, \theta_p^*)$  and  $v^* = (u_e^*, \theta_e^*)$  be the optimal controls used by the pursuer and the evader respectively. The Hamiltonian of the system satisfies the following conditions along the optimal trajectories[15]. These are called the *Isaacs* conditions.

1.  $H(\mathbf{x}, \nabla J, u, v^*) \leq H(\mathbf{x}, \nabla J, u^*, v^*) \leq H(\mathbf{x}, \nabla J, u^*, v)$
2.  $H(\mathbf{x}, \nabla J, u^*, v^*) = 0$

Condition 1 implies that when the players implement their optimal strategies any unilateral deviation by the pursuer leads to a smaller value for the Hamiltonian and

any unilateral deviation by the evader leads to a larger value of the Hamiltonian. Moreover condition 2 implies that when the players implement their optimal controls the Hamiltonian of the system is zero. The *Isaacs* conditions are an extension of the *Pontryagin's principle* in optimization to a game.

The Hamiltonian of our system is given by the following expression.

$$\begin{aligned} H(\mathbf{x}, \nabla J, u, v) &= \nabla J \cdot f(\mathbf{x}, u, v) + L \\ &= u_p [J_x \cos \theta_p - J_l \cos(\theta_p - \theta) - \frac{J_\theta}{l} \sin(\theta_p - \theta) + J_y \sin \theta_p] \\ &\quad + u_e [J_l \cos(\theta_e - \theta) + \frac{J_\theta}{l} \sin(\theta_e - \theta)] + 1 \end{aligned}$$

Since the evader wants to minimize the time of escape and the pursuer wants to maximize the time of escape, Isaacs first condition requires the following to be true along the optimal trajectories.

$$(u_e^*, \theta_e^*, u_p^*, \theta_p^*) = \min_{u_e, \theta_e} \max_{u_p, \theta_p} H(x, \nabla J, u, v) \quad (3)$$

We can see that the Hamiltonian is *separable* in the controls  $u_p$  and  $u_e$  i.e., it can be written in the form  $u_p f_1(\mathbf{x}, \nabla J) + u_e f_2(\mathbf{x}, \nabla J)$ . Hence the optimal controls for the players are given by the following expressions in terms of the gradient of the value function.

$$\begin{aligned} (\cos \theta_p^*, \sin \theta_p^*) &\parallel (J_x - J_l \cos \theta + \frac{J_\theta}{l} \sin \theta, J_y - J_l \sin \theta - \frac{J_\theta}{l} \cos \theta) \\ (\cos(\theta_e^* - \theta), \sin(\theta_e^* - \theta)) &\parallel (-J_l, -\frac{J_\theta}{l}) \\ u_e^* &= \bar{v}_e \\ u_p^* &= \bar{v}_p \end{aligned} \quad (4)$$

Due to lack of space, the derivation is presented elaborately in [9]. In the first and the second equation  $\parallel$  is used to denote parallel vectors. In case  $J_x - J_l \cos \theta + \frac{J_\theta}{l} \sin \theta = 0$  and  $J_y - J_l \sin \theta - \frac{J_\theta}{l} \cos \theta = 0$  then  $\theta_p^*$  can take any value and the pursuer can follow any control strategy. Similarly if  $J_l = 0$  and  $\frac{J_\theta}{l} = 0$ , then  $\theta_e^*$  can take any value and the evader can follow any control strategy. These conditions represent *singularity* in the Hamiltonian.

The entire game set can be partitioned into two regions depending on the value of the game. For all the initial positions of the pursuer and the evader for which the value of the game  $J(\mathbf{x})$  is finite, the evader can break the line of sight in finite time by following the strategies in Equation(4). For all the initial positions of the pursuer and the evader for which the value of the game is infinite, the pursuer can track the evader forever if it follows the controls given in Equation (4). Hence Equation (3) are the necessary and sufficient conditions for pursuer to track the evader in terms of the Hamiltonian of the system.

The analysis done in this section implies that if we are given the value function  $J(\mathbf{x})$  then we can compute the optimal strategies for the players by using equation (4).

## 5 Construction of Optimal Trajectories

In this section we present the trajectories generated by the optimal control laws that terminate on the UP. We use the following theorem to construct the optimal trajectories.

**Theorem 1.** [15] *Along the optimal trajectory, the following equation holds.*

$$\frac{d}{dt} \nabla J[\mathbf{x}(t)] = -\frac{\partial}{\partial \mathbf{x}} H(\mathbf{x}, \nabla J, u^*, v^*)$$

The above equation is called the adjoint equation and the components of  $\nabla J(\mathbf{x})$  are called adjoint variables. The retro-time (time-to-go) form of the adjoint equations is

$$\frac{d}{d\tau} \nabla J[\mathbf{x}(\tau)] = \frac{\partial}{\partial \mathbf{x}} H(\mathbf{x}, \nabla J, u^*, v^*)$$

where  $\tau = t_f - t$  is called the retro-time.  $t_f$  is the time of termination of the game.

The adjoint equation is a differential equation for the gradient of the value function  $J(\mathbf{x})$  along the optimal trajectories in terms of the optimal controls. Since Equation (4) gives the optimal controls of the players as a function of  $\nabla J(\mathbf{x})$ , we integrate the adjoint equations backward in time from the UP to obtain  $\nabla J(\mathbf{x})$  in terms of the state variables. Substituting  $\nabla J(\mathbf{x})$  into the optimal controls gives a feedback control strategy for the players. Substituting the feedback control laws for the players into the kinematic equation leads to the optimal trajectories. Due to lack of space, the construction of the optimal trajectories is provided elaborately in [9].

From the analysis done in [9], we present the optimal trajectories of the players. Let  $(x_f, y_f, l_f, \theta_f)$  denote the configuration of the bar at the termination situation. The optimal trajectory of the pursuer as a function of retro-time is given by the following equations.

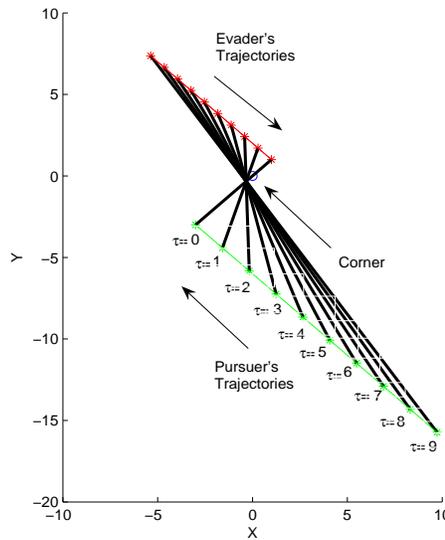
$$\begin{aligned} x_p(\tau) &= x_f + \tau \bar{v}_p \sin \theta_f \\ y_p(\tau) &= y_f - \tau \bar{v}_p \cos \theta_f \end{aligned} \quad (5)$$

The optimal trajectory of the evader as a function of retro-time is given by the following equations.

$$\begin{aligned} x_e(\tau) &= x_f + l_f \cos \theta_f - \bar{v}_e \tau \sin \theta_f \\ y_e(\tau) &= y_f + l_f \sin \theta_f + \bar{v}_e \tau \cos \theta_f \end{aligned} \quad (6)$$

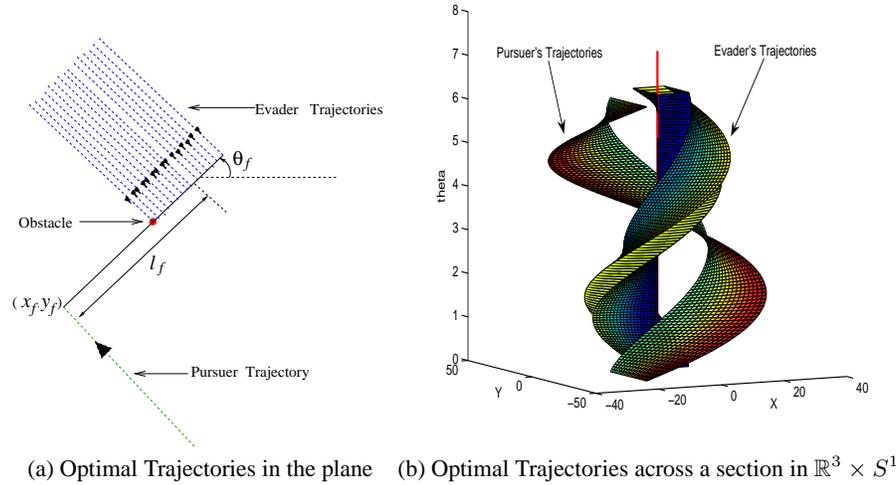
The optimal trajectories for the pursuer and the evader are straight lines. Moreover the trajectories are perpendicular to the orientation of the bar at the termination situation and hence parallel to each other. The players move in opposite directions as they follow the optimal trajectories. Figure 4 shows the optimal trajectories for the pursuer and the evader that terminate at a corner at the origin. The evader is shown by the red dots and the pursuer is shown by green dots. The black line joining the pursuer and the evader represents the orientation of the bar (line-of-sight) at different

time instants. The value of the speed ratio,  $r$ , is 0.5. At the termination situation, the bar is oriented at an angle of  $\frac{\pi}{4}$  with respect to the  $x$ , the position of the pursuer is  $(-3, -3)$  and the position of the evader is  $(1, 1)$ . The payoff for both the players at any point on the optimal trajectory is given by the variable  $\tau$  since it is the time required for termination. In the figure, the payoff for an orientation of the bar is shown on the side of the bar. The bar with  $\tau = 0$  represents the termination situation. If the pursuer deviates from its optimal strategy then the evader has a strategy for which it can escape around the corner in time less than  $\tau$ . If the evader deviates from its optimal trajectory then the pursuer has a strategy for which it can track the evader for a time greater than  $\tau$ . This is due to the fact that the trajectories are obtained from strategies that are in Nash equilibrium. Hence there is no motivation for either of the players to deviate from their optimal strategies.



**Fig. 4.** Optimal trajectories for a terminating situation around a corner

For a general environment in the plane, the optimal trajectories lie in  $\mathbb{R}^3 \times S^1$ . In order to depict them in  $\mathbb{R}^3$ , we need to consider a subspace of the optimal paths terminating at a corner. In the following examples, for each corner in the environment we show the subspace of the optimal paths that have a fixed distance of the pursuer from the corner at the termination situation. The value of the speed ratio,  $r$ , is 0.66 in all the following examples. Figure 5 shows the optimal trajectories for the players in a simple environment containing a point obstacle at the origin. The line of sight between the pursuer and the evader is broken if it passes through the origin. The evader wants to minimize the time required to break the line of sight and the pursuer

**Fig. 5.** Optimal trajectories for an environment having a single point obstacle

wants to maximize it. Let  $(x_f, y_f, l_f, \theta_f)$  represent the orientation of the bar at the termination situation. Figure 5(a) shows the optimal trajectories of the players for all possible values of  $l_f$  for a constant value of  $x_f, y_f$  and  $\theta_f$ . Figure 5(b) shows the optimal trajectories for every orientation of the bar at the termination situation. The  $z$  axis represents the angle of the bar at the termination situation. A cross-section parallel to the  $xy$ -plane gives the optimal trajectories of the players in a plane for a given  $\theta_f$ . The red line in the middle denotes the point obstacle. The inner spiral is formed by the optimal trajectories of the evader and the outer spiral is formed by the optimal trajectory of the pursuer. The color of a point is a representative of the value of the game,  $J(\mathbf{x})$ , at that point. The value of the game increases as the color changes from blue to red.

Figure 6(a) shows a single corner in the plane. The internal angle at the corner is  $\frac{2\pi}{3}$ . Figure 6(b) shows the optimal trajectories of the players for the corner. The symmetry in the trajectories is due to the fact that termination situations occur symmetrically around a corner.

Figure 7(a) shows a regular hexagon in the plane. Figure 7(b) shows the optimal trajectories of the players for the hexagonal obstacle.

## 6 Conclusion and Future work

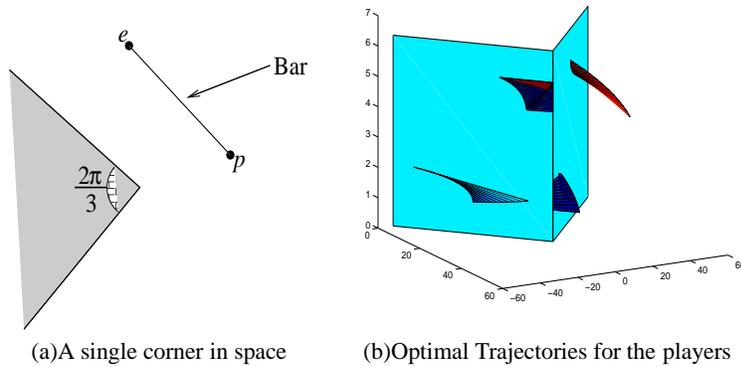
In this paper, we address a visibility based pursuit-evasion game in an environment containing obstacles. The pursuer and the evader are holonomic having bounded speeds. The pursuer wants to maintain visibility of the evader for maximum possible

time and the evader wants to escape the pursuer’s sight as soon as possible. Both the players have knowledge about each others current position. Under this information structure, we present necessary and sufficient conditions for surveillance and escape. We present strategies for the players that are in Nash Equilibrium. The strategies are a function of the value of the game. Using the strategies, we construct a value function by backward integration of the adjoint equations from the termination situations provided by the corners in the environment. From the value functions we recompute the control strategies for the players to obtain optimal trajectories for the players near the termination situation. We show that the optimal strategy for the players is to move on straight lines parallel to each other in opposite directions towards a termination situation. We show a subspace of the optimal trajectories for a point obstacle, a corner and a hexagonal obstacle in space.

In order to extend the results in this paper to environment containing multiple obstacles, we plan to address the following issues in the future.

1. In a general environment there might be points from which the pursuer can see the entire free space. Such environments are called *star-shaped* and the set of points for which the property holds is called the *kernel* of the star-shaped environment. If the pursuer can reach the kernel while keeping the evader in its sight then the pursuer can see the evader forever. Hence the kernel also provides a termination situation where the pursuer can track the evader forever. The shape of the kernel depends on the shape of the environment. At this moment, we do not have a general characterization of the shape of the kernel in the configuration space i.e.,  $\mathbb{R}^3 \times S^1$  which makes it difficult to compute the final conditions for the adjoint variables in the adjoint equation. Hence we are unable to present the optimal trajectories that are generated back from the termination situation posed due to the presence of such regions.

Fig. 6. Optimal trajectories of the players for a corner in space



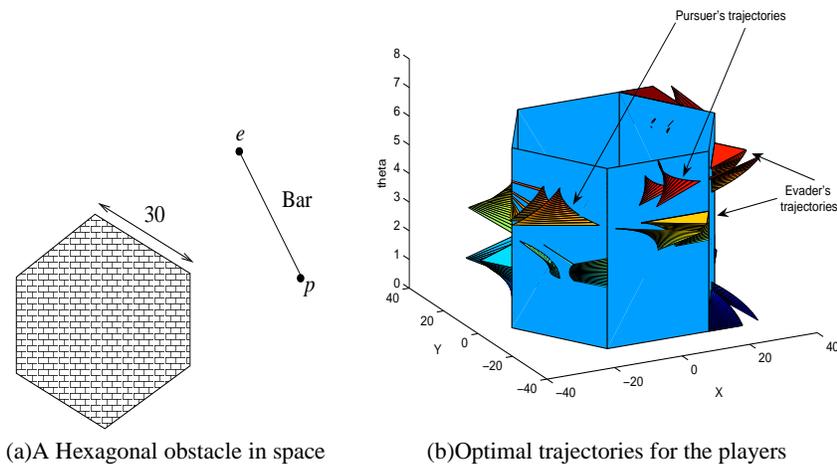
- In the previous section, we presented the optimal trajectories for the players terminating at a corner in the environment. Consider a point  $\mathbf{x}$  in the state-space that lies on two different optimal trajectories terminating at different corners in the environment. Figure 8 shows such a situation. Both the figures show the optimal trajectories for two regular hexagonal obstacles in the environment. Figure 8(a) shows points in the space that lie on two different optimal trajectories for the evader. Similarly, Figure 8(b) shows points in the space that lie on two different optimal trajectories for the pursuer. Each optimal trajectory will assign a value,  $J(\mathbf{x})$ . In case the two value functions are different at  $\mathbf{x}$ , we need to modify the analysis since the value of the game at  $\mathbf{x}$  is unique. Moreover, we also need to ensure that  $\nabla J(\mathbf{x})$  is well defined for a *regular* construction or else we need to look for *viscosity* solutions.

In the future, we plan to use *viscosity* solutions to propose an algorithm to construct the optimal trajectories for the players in a environment containing multiple obstacles. We also plan to extend the results to multiple pursuers chasing an evader. We also plan to extend our work to players having non-holonomic constraints in their motion.

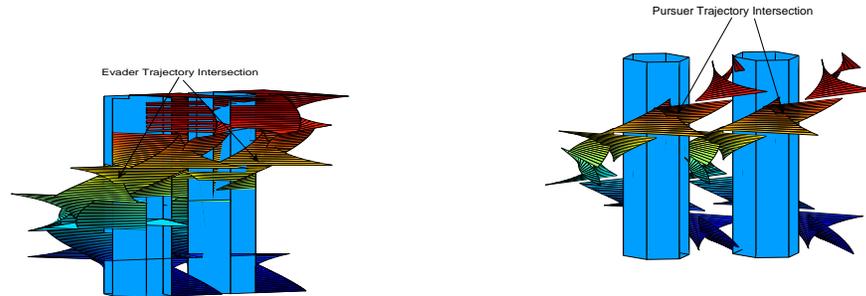
## References

- S. Alexander, R. Bishop, and R. Ghrist. Pursuit and evasion in non-convex domains of arbitrary dimensions. In *Proceedings of Robotics: Science and Systems*, Philadelphia, USA, August 2006.
- S. Alexander, R. Bishop, and R. Ghrist. Capture pursuit games on unbounded domains. 2008.

**Fig. 7.** Optimal trajectories of the players for a hexagonal obstacle in space



**Fig. 8.** Intersection of the optimal trajectories of the evader in an environment containing 2 hexagonal obstacles



(a) Intersection of the optimal trajectories of the evader (b) Intersection of the optimal trajectories of the pursuer

3. T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory, 2nd Ed.* SIAM Series in Classics in Applied Mathematics, Philadelphia, 1999.
4. T. Bandyopadhyay, Y. Li, M. Ang Jr, and D. Hsu. Stealth Tracking of an Unpredictable Target among Obstacles. *Proceedings of the International Workshop on the Algorithmic Foundations of Robotics*, 2004.
5. T. Bandyopadhyay, Y. Li, M. Ang Jr., and D. Hsu. A Greedy Strategy for Tracking a locally Predictable Target among Obstacles. *Robotics and Automation, Proceedings. ICRA'02. IEEE International Conference on*, pages 2342–2347, 2006.
6. C. Becker, H. Gonzalez-Banos, J. Latombe, and C. Tomasi. An intelligent observer. *Proceedings of International Symposium on Experimental Robotics*, pages 94–99, 1995.
7. S. Bhattacharya, S. Candido, and S. Hutchinson. Motion strategies for surveillance. In *Robotics: Science and Systems - III*, 2007.
8. S. Bhattacharya and S. Hutchinson. Approximation schemes for two-player pursuit evasion games with visibility constraints. In *Proceedings of Robotics: Science and Systems IV*, Zurich, Switzerland, June 2008.
9. S. Bhattacharya and S. Hutchinson. From strategies to trajectories. <http://www-cvr.ai.uiuc.edu/~sbhatac/comp.pdf>, 2008.
10. A. Efrat, H. Gonzalez-Banos, S. Kobourov, and L. Palaniappan. Optimal strategies to track and capture a predictable target. *Robotics and Automation, 2003. Proceedings. ICRA'03. IEEE International Conference on*, 3, 2003.
11. P. Fabiani and J. Latombe. Tracking a partially predictable object with uncertainty and visibility constraints: a game-theoretic approach. Technical report, Technical report, University of Stanford, December 1998. <http://underdog.stanford.edu/>.(cited on page 76).
12. W. H. Fleming. The convergence problem for differential games. *Journal for Mathematical Analysis and Applications*, 3:102–116, 1961.
13. W. H. Fleming. The convergence problem for differential games. *Advances in Game Theory. Annals of Mathematics Studies*, 52:195–210, 1964.

14. H. Gonzalez-Banos, C. Lee, and J. Latombe. Real-time combinatorial tracking of a target moving unpredictably among obstacles. *Robotics and Automation, 2002. Proceedings. ICRA'02. IEEE International Conference on*, 2, 2002.
15. R. Isaacs. *Differential Games*. Wiley, New York, 1965.
16. S. Kopparty and C. V. Ravishankar. A framework for pursuit evasion games in rn. *Inf. Process. Lett.*, 96(3):114–122, 2005.
17. S. M. LaValle, H. H. Gonzalez-Banos, C. Becker, and J. C. Latombe. Motion strategies for maintaining visibility of a moving target. In *Robotics and Automation, 1997. Proceedings., 1997 IEEE International Conference on*, volume 1, pages 731–736, Albuquerque, NM, USA, Apr. 1997.
18. J. Lewin. *Differential Games: Theory and Methods for Solving Game Problems with Singular Surfaces*. Springer-Verlag, London, 1994.
19. T. Li, J. Lien, S. Chiu, and T. Yu. Automatically generating virtual guided tours. *Computer Animation Conference*, pages 99–106, 1997.
20. A. A. Melikyan and N. V. Hovakimyan. Game problem of simple pursuit on a two-dimensional cone. *Journal of Applied Mathematics and Mechanics*, 55(5):607–618, 1991.
21. A. A. Melikyan and N. V. Hovakimyan. Singular trajectories in the game of simple pursuit in the manifold. *Journal of Applied Mathematics and Mechanics*, 55(1):42–48, 1991.
22. A. A. Melikyan and N. V. Hovakimyan. A differential game of simple approach in melikyan. *Journal of Applied Mathematics and Mechanics*, 57(1):47–57, 1993.
23. R. Murrieta-Cid, T. Muppirlala, A. Sarmiento, S. Bhattacharya, and S. Hutchinson. Surveillance strategies for a pursuer with finite sensor range. *International Journal of Robotics Research*, pages 1548–1553, 2007.
24. T. Parsons. Pursuit-evasion in a graph. *Theor. Appl. Graphs, Proc. Kalamazoo*, 1976.