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Controllability and Properties of Optimal paths for a Differential Drive robot with field-of-view constraints

Sourabh Bhattacharya **Seth Hutchinson**
Department of Electrical and Computer Engineering
Beckman Institute for Advanced Science and Technology
University of Illinois at Urbana-Champaign
{ sbhattac,seth } @ uiuc.edu

Abstract [illegible]

I. Introduction

In this paper, we study the interaction of the nonholonomic and visibility constraints for a robot that maintains visibility of a stationary landmark. The robot is a differential drive system and has limited perception (angle of view). We first demonstrate controllability of the resulting system, and then describe the properties of optimal paths for the system.

The study of optimal paths for nonholonomic systems has been addressed by numerous researchers (a nice overview is given in [5]). Dubins [4] determined the shortest paths for a car-like robot than can only go forward. Reeds and Shepp extended this work and established the shortest length paths for a car-like robot that can move forward and backward [7]. Balkcom and Mason determined the time-optimal trajectories for a differential drive robot [1]. All of these results assume that the nonholonomic robot moves in the free space (without obstacles). These previous results do not address the case with sensing constraints on the robot. In this paper we address the combination of nonholonomic constraints and constraints imposed by the sensor. The latter essentially define a forbidden region in the configuration space of the system. In [3], the controllability of the system has been analysed with constraints in the viewing range and optimal paths were proposed for the mode in which the robot cannot head directly towards the target. In this paper we analyze the system without the range constraint and the robot is allowed to move directly towards the target. We give a more elegant proof of controllability of the system. We demonstrate that optimal paths consist of segments that are either straight lines in the plane or curves that saturate the sensor viewing angle, and propose the properties of optimal paths.

II. Problem Definition

We make the usual assignment of body-attached frame to the robot, with origin at the midpoint between the two wheels, y-axis parallel to the axle, and the x-axis pointing forward, parallel to the heading of the robot. The configuration of the robot can be represented by $(x, y, \psi)^T$, in which ψ is the angle from the world x-axis to the robot's x-axis.

The heading of the robot is defined as the direction in which the robot moves. Since a differential drive robot can move forward and backward at a point, the heading angle with respect to the robot's x-axis is zero or π.

We can also use polar coordinates to represent the position of the center of the robot in the Cartesian plane by introducing the following transformations:

$$r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\frac{y}{x} \qquad (1)$$

In polar coordinates the configuration of the robot can be represented by $(r, \theta, \psi)^T$. The camera is positioned so that the optical center lies directly above the origin of the robot's local coordinate frame. The optical axis is parallel to the world x-y plane, and the pan angle ϕ is the angle from the robot's x-axis to the optical axis. We assume that the range of camera rotation is limited, such that $\phi \in [\phi_1, \phi_2]$. Without loss of generality, we place the (static) landmark at the origin of the world coordinate system. These conventions are illustrated in Figure 1.

Given this formulation, the problem that we consider is that of finding minimal length paths from initial to goal position (without regard to the robot orientation) such that the following conditions are satisfied:
1. The camera is always pointing toward the land-

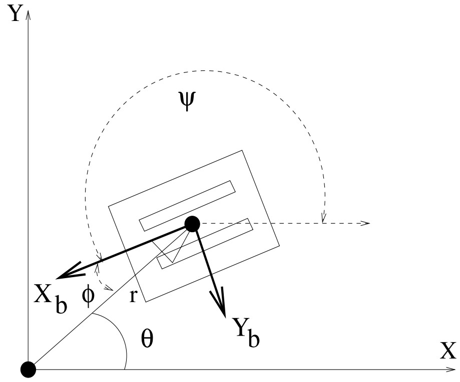

Fig. 1. Coordinate frame assignments for a differential drive robot with camera.

mark, i.e.,

$$\psi + \phi = \pi + \tan^{-1} \frac{y}{x} \tag{2}$$

2. Constraints on camera motion are not violated, i.e.,

$$\phi \in [\phi_1, \phi_2] \tag{3}$$

and also $0 \in [\phi_1, \phi_2]$, so that the robot can move directly towards the target.

III. Controllability

In this section we address the issue of controllability of our system. Controllability of a system is defined as its ability to reach any admissible goal configuration in its configuration space from any admissible initial configuration in its configuration space using a bounded control in finite time. Controllability of our system is important for two reasons

1. Optimal paths between any two given configurations in the state space can be obtained only when the system can be driven from one configuration to the other using bounded control input.
2. The paths used to prove the controllability of the system in a constructive way also help us in finding the nature of optimal paths.

Next we derive the equations of curves that maintain a constant angle between the line of sight and the direction of motion of the DDR. We then use these curves in our proof of controllability.

IV. T Curves

Consider the curve traced out by the DDR (differential drive robot) through a generic point (r_0, θ_0, ψ), respecting the constraint that the angle between the heading of DDR and the optical axis of the camera is held at a constant ϕ. The optical axis of the camera should always be pointing toward the landmark, located at the center of the circle. We refer to such a curve as a T curve.

Proposition 1 : The equation of the T curve passing through the point (r_0, θ_0) with fixed pan angle ϕ is given by

$$r = r_0 e^{\frac{(\theta_0 - \theta)}{\tan \phi}} \tag{4}$$

Proof Refer to figure 2. Let (r_0, θ_0) be the coordinates of the point through which the curve passes. Let (x, y) be the coordinates of a general point on the curve. From [6], the differential equation of the curve

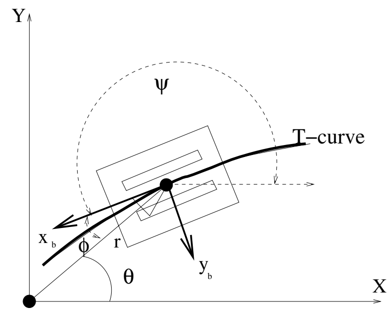

Fig. 2. T-curve construction

traced by the DDR is

$$\frac{1}{r}\frac{dr}{d\theta} = -\cot \phi$$

$$\frac{1}{r} dr = -\cot \phi d\theta$$

Integrating on both sides

$$\int_{r_0}^{r} \frac{1}{r} dr = \int_{\theta_0}^{\theta} -\cot \phi d\theta$$

Since ϕ is a constant

$$\ln \frac{r}{r_0} = (\theta_0 - \theta) \cot \phi \tag{5}$$

On simplification we obtain

$$r = r_0 e^{\frac{(\theta_0 - \theta)}{\tan \phi}} \tag{6}$$

∎

Since ϕ is allowed to take values in $[\phi_1, \phi_2]$, two curves can be drawn through any point in the x-y plane such that ϕ takes the values at the extremities of the interval. The space between the two curves represents the possible heading directions of the robot that satisfy the visibility constraints. Hence the curves can be thought of as latitudes and longitudes. Each such curve can be put into one of two categories as shown in Figure 3:

1. T1: These are the curves that maintain a relative angle of ϕ_1 between the optical axis of the camera and the heading angle of the robot.

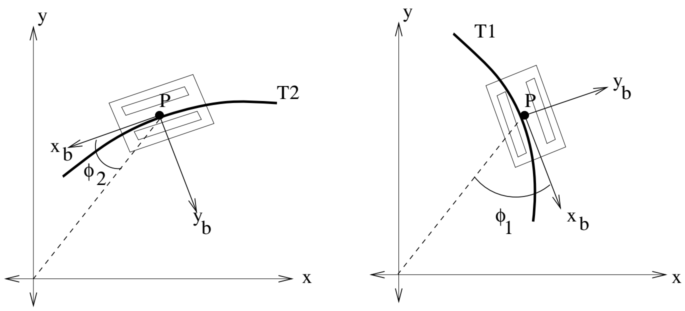

Fig. 3. A T1 and T2 curve passing through P.

2. T2: These are the curves that maintain a relative angle of ϕ_2 between the optical axis of the camera and the heading angle of the robot.
If $-\pi/2 < \phi_i < 0$, increasing θ causes r to grow. If $\pi/2 > \phi_i > 0$, increasing θ causes r to decrease. Hence when $-\pi/2 < \phi_1 < 0$ and $\pi/2 > \phi_2 > 0$, we can spiral out by following the T1 curve and spiral in by following the T2 curve in an anticlockwise sense. Or we can spiral in by following the T1 curve and spiral out by following T2 curve in a clockwise sense.

V. Constructive Proof of Controllability

The following property of two T-curves is essential in proving the controllability.
Property 1: If $\phi_2 - \phi_1 \neq \pi$, any T1 curve intersects any T2 curve at a point located at a finite distance from the origin.

Proof Let P= (r_1, θ_1) and Q= (r_2, θ_2) be any two points in the plane. By proposition 1, the equation of a T1 curve passing through P is

$$r = r_1 e^{\frac{(\theta_1 - \theta)}{\tan \phi_1}} \tag{7}$$

The equation of a T2 curve passing through Q is

$$r = r_2 e^{\frac{(\theta_2 - \theta)}{\tan \phi_2}} \tag{8}$$

Taking the natural logarithm of both sides of Equations (6) and (7) and on further subtraction and simplification we obtain

$$r = r_1^{\frac{\tan \phi_1}{(\tan \phi_1 - \tan \phi_2)}} r_2^{\frac{\tan \phi_2}{(\tan \phi_1 - \tan \phi_2)}} e^{\frac{\theta_1 - \theta_2}{(\tan \phi_1 - \tan \phi_2)}} \tag{9}$$

Since $\phi_2 - \phi_1 \neq \pi$ and $\phi_1 \neq \phi_2$, r is finite.
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Proposition 2: The DDR is controllable between any two points in the configuration space.

Proof Let P and Q be two points in the work space. Let R be the intersection of the T1 curve from P and T2 curve from Q. Let the DDR start at P in any configuration satisfying the constraint Equations (3) and (4). Now rotate the DDR at P so that the angle between the robot's x-axis and the radius vector, ϕ, becomes ϕ_1. Now move along the T1 curve until the DDR reaches R. At R, the DDR rotates so that ϕ increases from ϕ_1 to ϕ_2. Now the DDR moves on the T2 curve to reach Q. By proposition 1, if $\phi_2 - \phi_1 \neq \pi$ then R exists. Since the choice of P and Q is arbitrary, the system is controllable in the case $\phi_2 - \phi_1 \neq \pi$.
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VI. Optimality of Paths

Since now the system has been shown to be controllable, we can move on to the next step of finding the properties of optimal paths. If we denote by $T1_P$ and $T2_P$ the $T1$ and $T2$ curve, respectively, through the point P, we can see that $T1_P$ and $T2_P$ divide the plane around P into four disjoint regions irrespective of the numerical values of ϕ_1 and ϕ_2. We have followed a nomenclature of naming those regions as shown in Figure 4. The line from the target to the point P passes through two of these regions. One of those regions contains the target and is called the D type region and the other region is called the C type region. The remaining two regions are given the names A and B as shown in Figure 4. Since the range of the viewing angle includes the direction of motion, then the possible heading of the robot from P can only be in the region C or D.

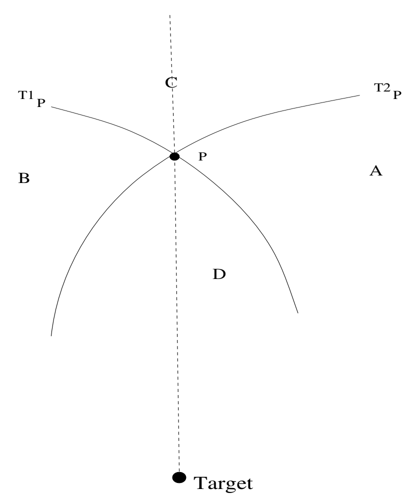

Fig. 4. State space division around P by T curves.

VII. Properties of Optimal Paths

Let P be the initial point and O be the target as shown in Figure 5. The line PO divides the plane into two half planes, denoted by P^+ and P^-. Let Q be the goal point.
Property 1: An optimal path from P to Q never crosses the line PO. The optimal path from P to any point Q on the line PO is the straight line PQ.

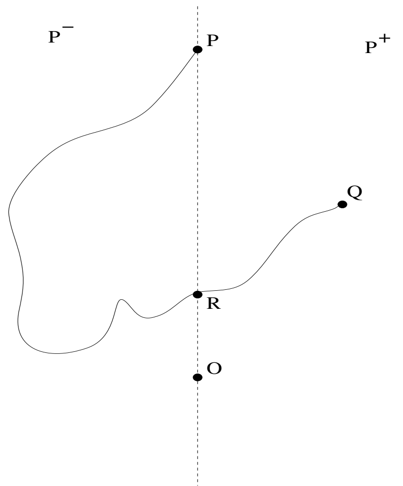

Fig. 5. P is the initial point. O is the target. The line PO divides the plane into two half planes P^+ and P^-. Q is the goal point.

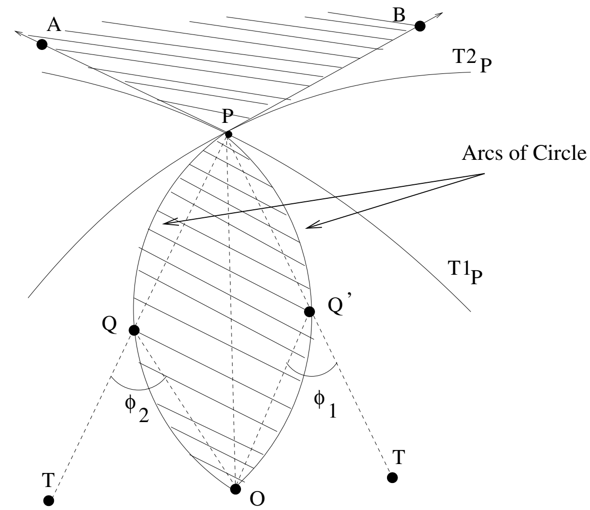

Fig. 6. S-set.

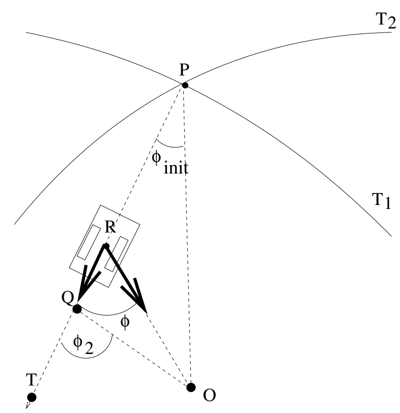

Fig. 7. S-set derivation for region of type D.

Proof Refer again to Figure 5. Suppose the optimal path L intersects the line PO at a point R. If L is optimal then the part of the path from P to R on L also must be optimal. But the optimal path from P to any point R on this line is the straight line from P to R, the DDR can move straight toward and away from the target. Hence if Q does not lie on the line PO, L can never intersect the line PO. If Q lies on line PO, then the optimal path from P to Q is the straight line PQ.
■

We define the S-set of a point P to be the set of points that can be reached on a straight line path from P. The next two properties comprise a derivation for the shape of S-set.

Property 2: If the robot heading at P points into a region of type D, then the S-set is the region bounded by the arc of circles tangent to $T1_P$ and $T2_P$ at P and passing through the origin O. This is shown in Figure 6. Hence at P, $T2_P$ and the arc OQP share the same tangent, and $T1_P$ and the arc OQ'P share the same tangent.

Proof Refer to Figure 7. Since the robot is heading into a region of type D, the angle between its heading and the radius vector is given by ϕ. Let the robot start heading on a straight line from P in a direction such that $\phi \in [0, \phi_2]$. Let R denote the position of the robot as it moves forward along the line PT. Then ϕ is $\angle$ORT. It can be seen from the figure that as length PR increases, $\angle$ORT increases. Hence the robot can move only until it reaches the point R=Q such that $\angle$OQT= ϕ_2. This is true for any $\phi \in [0, \phi_2]$. Hence the end point Q on PT satisfies the constraint $\angle$OQT= ϕ_2. From theorems of plane geometry regarding circles, we conclude that the locus of point Q is an arc of a circle circumscribing $\triangle$PQO and the tangent at P to the arc OQP makes an angle of ϕ_2 with the line segment PO, which is the same as the tangent to the T2 curve at P. For further reference refer to [2]. Hence, arc OQP and T2$_P$ share the same tangent at P. Similarly, if $\phi \in [\phi_1, 0]$ the locus of Q' is the arc of the circle circumscribing $\triangle$PQ'O such that $\angle$OQ'T= ϕ_1 and the arc OQ'P and T2$_P$ share the same tangent at P. Hence the S-set in this case is the union of sectors of two circles, as shown in Figure 6.
■

Property 3: If the robot heading from P is in the region of type C, then the S-set is the region bounded by the lines tangent to T1$_P$ and T2$_P$ at P. This is illustrated in Figure 6.

Proof Refer to Figure 8. Let the robot start heading on a straight line in a region of type C from P, in a direction such that ∠OPT= α. Let R denote the position of the robot as it moves forward along the line PT. As the robot moves ahead along PT, ∠ORT increases if $\alpha \leq \pi$ and decreases if $\alpha \geq \pi$. As the robot moves to infinity on line PT, α tends to π, which in turn implies that ϕ tends to zero, which satisfies the constraints of the DDR. Hence there is no constraint on the limit to which the robot can move. Hence the S-set in this case is the region enclosed by the rays PA and PB as shown in Figure 6.

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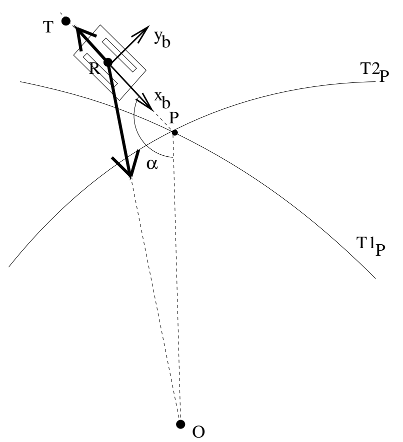

Fig. 8. S-set derivation for region of type C.

The next property shows that the optimal paths are obtained by joining the T-curves and the straight lines.
Property 4: Optimal paths consist of straight line segments and sections of T-curves.

Proof Consider a point P on the optimal path, as illustrated in figure 9 . In the neighbourhood of P, the path either passes through the S-set of P, in which case the path is a straight line, or it is tangent to the S-set, in which case it is $T1_P$ or $T2_P$ by Property 2 and Property 3. Hence, optimal paths are composed of segments of T1 and T2 curves, straight lines, and their combinations.

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Based on property 4 we can enumerate the kinds of non-differentiable transitions allowed. Refer to Figure 10. Let the optimal path be denoted by L. Take a ball B_ϵ of radius ϵ around a nondifferentiable point P on L where ϵ is chosen small so that P is the only nondifferentiable point on L in the ball. Consider the part of L inside B_ϵ. This can be denoted as $B_\epsilon \cap L$. Now P divides $B_\epsilon \cap L$ into two smaller segements, L_1 and L_2. Hence $B_\epsilon \cap L - \{P\} = L_1 \cup L_2$. We only consider continuous paths in which the set of non-differentiable points has measure zero. Hence we can

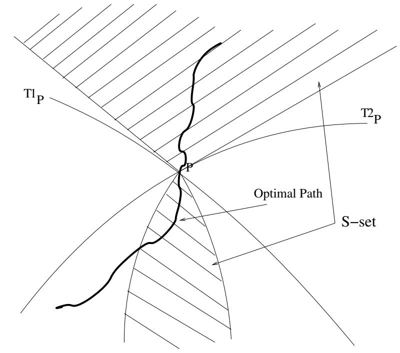

Fig. 9. Nature of continuous paths for

find an $\epsilon > 0$ such that $L_1, L_2 \in C^1$. Since L_1 and L_2 belong to C^1, by Property 6, if L is optimal, each of L_1 and L_2 is a straight line or a T-curve. Hence we can reduce the family of optimal paths through P to be of the forms as shown in Figure 11. In Figure 11, the notations stand for

- SL_C - Straight line in region of type C.
- SL_D - Straight line in region of type D.
- $T2_{AC}$ - T2 curve at the common boundary of regions A and C.
- $T1_{AD}$ - T1 curve at the common boundary of regions A and D.
- $T1_{BC}$ - T1 curve at the common boundary of regions B and C.
- $T2_{BD}$ - T2 curve at the common boundary of regions B and D.

Therefore the problem has been reduced to eliminating those cases in which the path can be shortened, respecting the kinematic constraints of the DDR.

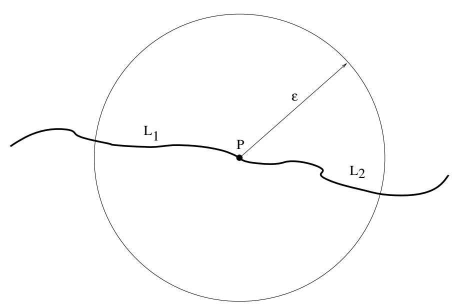

Fig. 10. Optimal path through point P.

From Figure 11, we can enumerate the following cases:

1	$SL_C - T1_{AD}$	9	$T2_{AC} - SL_D$
2	$SL_C - T1_{BC}$	10	$T1_{BC} - T1_{AD}$
3	$SL_C - T2_{AC}$	11	$T2_{BD} - T2_{AC}$
4	$SL_C - T2_{BD}$	12	$T1_{AD}T2_{AC}$
5	$SL_C - SL_D$	13	$T2_{BD}T1_{BC}$
6	$T1_{BC} - SL_D$	14	$T1_{BC}T2_{AC}$
7	$T1_{AD} - SL_D$	15	$T2_{BD}T1_{AD}$
8	$T2_{BD} - SL_D$		

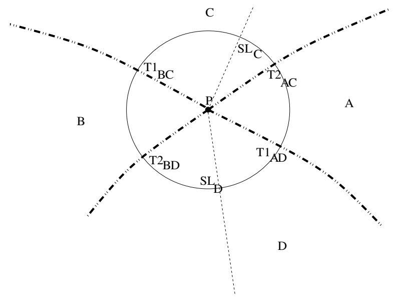

Fig. 11. Types of optimal paths around a nondifferentiable point.

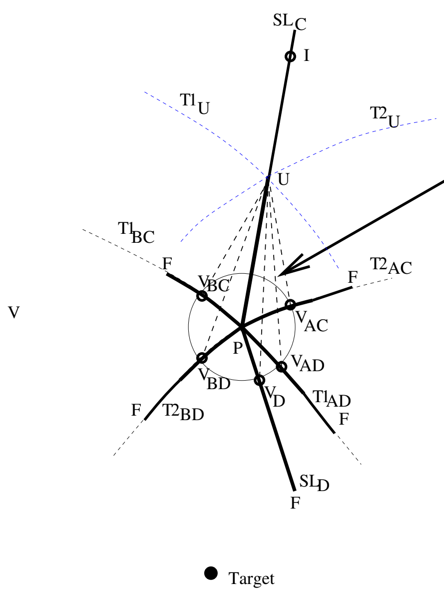

Fig. 12. Shorter paths for cases 1 to 5.

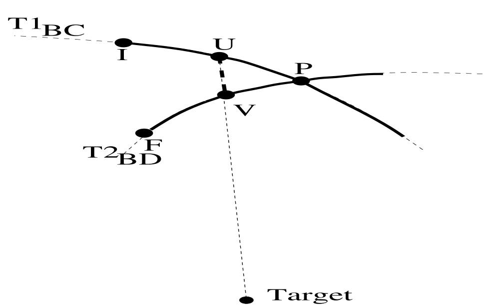

Fig. 13. Shorter paths for cases 12 and 13.

In Figures 12 and 13, $I, F \in B_\epsilon$ are the initial and final points of a path segment under consideration. We first describe the shortening of the paths for cases 1 to 5. Consider Figure 12. Consider a point U on SL_C. Since there is a straight line from U to P, P lies in the S-set of U. Since the S-set for a point U includes an open set bounded by the arcs of two circles (recall Figure 6) for some U very close to P, there is a neighbourhood of P, B_P, that lies in the S-set of U. B_P intersects with $T1_{AD}$, $T1_{BC}$, $T2_{AC}$, $T2_{BD}$, and SL_D at V_{AD}, V_{BC}, V_{AC}, V_{BD}, and V_D respectively. Since V_{AD}, V_{BC}, V_{AC}, V_{BD} and V_D lie in the S-set of U, straight lines can be drawn from U to to each of them and this shortens the path for each of the above cases. The shortening of the path is shown by the dashed lines. Cases 6 to 9 can be shortened in the same manner. For cases 10 and 11, P is not a nondifferentiable point. Now let us consider the cases 12 and 13. Consider the path $T2_{BD}T1_{BC}$. Refer to Figure 13. In DTS mode, the DDR can move on a straight line from the point U on $T1_{BC}$ to the origin and this intersects $T2_{BD}$ at the point V. The dashed line UV shortens the path $T2_{BD}T1_{BC}$. The same arguments can be applied to $T1_{AD}T2_{AC}$. The only cases that remain are that of $T1_{BC}T2_{AC}$ and $T2_{BD}T1_{AD}$. Hence for C^0 paths to be optimal, any nondifferentiable point has to be of the above type.

VIII. Conclusion

In this paper we show that the system is controllable. We present the properties of optimal paths for this system. We also reduce the possible kinds of nondifferentiable transitions on an optimal path from fifteen possible cases to two.

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