Hints and Notes for Homework 7

problems 5-1 thru 5-5

If these seem simple and straightforward, that's because they are. Your answer to each problem should be in the form of a single line

$$Q = \text{configuration space}$$

where the configuration space is some combination of $\mathbb{R}^n$ and/or T^m (the m-torus).

If these seem confusing and complicated, it's probably due to some confusion between the concepts of configuration space and workspace or between different notations for revolute spaces. Here's a primer on configuration space:

- **Note 1**: *configuration space* and *workspace* are entirely different animals. For example, consider the illustrations of workspaces in Figure 1.17 on page 18 of the text. Robots (a) and (b) have identical configuration spaces ($Q = T^2 \times \mathbb{R}^1$) while the figures show that their workspaces have different shapes. Interestingly, the cartesian robot of (d) has configuration space $Q = \mathbb{R}^3$ which happens to be exactly the same as its workspace, also $\mathbb{R}^3$. However, imagine adding three more prismatic joints to the cartesian robot; this 6-jointed robot would have configuration space $Q = \mathbb{R}^6$, while its workspace would remain the same as before, $\mathbb{R}^3$.

- **Note 2**: There is some confusion when it comes to notation for revolute spaces. Note that T^1, the 1-torus, is the same as S^1, the unit circle, is the same as $SO(2)$, the group of all rotations in the plane.

  $$T^1 = S^1 = SO(2)$$

  Confusingly, T^2, the 2-torus or doughnut, is NOT the same as S^2, the unit sphere, and neither are the same as $SO(3)$, the group of all 3D rotations. While it is true that

  $$T^1 \times T^1 = T^2$$

  it is NOT true that $S^1 \times S^1 = S^2$, NOR is it the case that $SO(2) \times SO(2) = SO(3)$. The correct description of the configuration space of a robot with two revolute joints is T^2, NOT S^2 NOR $SO(3)$. I suggest sticking with $\mathbb{R}^n$ and T^m notations for this problem.

problem 5-6

The problem wording might be a little confusing. There are 6 links total in the arm; the spherical wrist is the last 3 links of the 6-link anthropomorphic arm. All joints are revolute.

problem 5-7

Straightforward. Set up the equations, apply the gradient.

problem 5-8 thru 5-9 clarification

The words "planar" and "polygon" are causing confusion, and understandably so. Technically, all polygons are planar (see definition on Mathworld). However, Hutchinson intends that "polygon" in 5-10 be treated as a non-planar structure.

There is some debate about whether the point p in problem 5-9 lies in the same plane as the polygon. I think the "plane" part of this problem statement is meant to differentiate the planar "polygon" in problem 5-9 from the non-planar "polygon" in problem 5-10. As such, I would like you to consider the case where the point p is not necessarily in the same plane as the polygon.

Here's the summary:

- **problem 5-8**: Since three points define a plane, this problem can only be considered as planar.
- **problem 5-9**: The word "polygon" here should be read to mean a planar shape with straight edges and a single face. The point p, however, should not be assumed to lie in the same plane as the face of the polygon.
- **problem 5-10**: The word "polygon" here should be read to mean "polyhedron," a shape with straight edges and multiple faces. Again, the point p should not be assumed to lie in the same plane as any of the faces of the "polygon."

problem 5-8

You are computing the minimum distance from a point to a line *segment*, not a line. This give rises to two possible cases, depending on whether the perpendicular from point p intersects the line segment. The following notation may be helpful:

Let A denote the line *segment* passing through a_1 and a_2.
Let P be the *line* passing through p that is perpendicular to A.

There are two cases to consider:

- *case 1*: P intersects A. In this case, you should report the perpendicular distance from p to A. It will be helpful to compute the point $a_\perp$ where P intersects A. Let

  $$a_\perp = a_1 + t_\perp \cdot (a_2 - a_1)$$

  for some $t_\perp \in \mathbb{R}$. You can compute $t_\perp$ by using the fact that the dot product of A and P is zero. Once you know $t_\perp$, you know where $a_\perp$ is. (There are other ways of finding $a_\perp$ than the method I have elaborated on here. You need not use my method, however YOUR SOLUTION MUST GIVE AN EXPLICT EXPRESSION FOR $a_\perp$ TO RECEIVE FULL CREDIT) Once $a_\perp$ is found, the minimum distance between p and A is simply

  $$\|a_\perp - p\|.$$

  case 2: P does not intersect A. In this case, the minimum distance from p to A is the smaller of the distances from p to a_1 and p to a_2. That is, the minimum distance is

  $$min\{\|p - a_1\|, \|p - a_2\|\}.$$

problem 5-9

This problem is complicated! First of all, do not assume that p lies in the same plane as the polygon. Proceed according to the following algorithm:

1. compute the distances from p to each of the *vertices*.
2. compute the perpendicular distances for p to each of the *edges*.
3. compute the perpendicular distance from p to the *face* of the polygon.

The minimum distance from p to the polygon will be the minimum value reported by Parts 1 through 3. Part 1 is straightforward. For Part 2, simply invoke the result of problem 5-8. For part 3, we must

- compute the normal vector to the planar face of the polygon. You may assume that at least three of the vertices are not collinear; call them v_1, v_2, v_3. Give a valid expression for the normal vector, n. (Hint: use the cross-product).
- compute the equation of the plane itself. All points in the plane will satisfy an equation of the form $ax + by + cz = d$, and the vector $[a, b, c]$ which is equivalent to the normal vector n. Find the equation of the plane.

Now solve for the point of intersection ℓ of the plane with the line defined by the normal n that passes through point p.

Finally, we must check to see whether ℓ is inside or outside the face of the polygon. To do this, we can draw lines L_i from ℓ to each of the vertices a_i of the polygon. Now, select any line in the plane to use as a angular reference. If the sum of the angles made by the lines L_i with the reference line is an integer multiple of 2π, the intersection point ℓ lies inside the face. If the point is inside the face of the polygon, we record the distance from the point p to ℓ. (for this part of the problem–the deduction of whether the intersection lies inside the polygon–it is not necessary that you show all the math; I will be satisfied if you simply copy the text of this procedure)

problem 5-10

Assume the 3D polygon can be decomposed into m flat faces G_i.

problem 5-11

$\rho(o_i(q))$ may look scary and complicated, but it's really harmless. Distance $\rho(o_i(q))$ is a function of the location in space $o_i(q)$. Consider writing this as $\rho(x)$, where x is a vector in three dimensions. Thus, we can treat the gradient ∇ as the partial derivative w.r.t. the vector x. The result now follows from the chain rule.

problem 5-12

I strongly recommend drawing a picture! You may make the following assumptions:

1. the repulsive forces act *only* on the four vertices of the robot.
2. the robot has three degrees of freedom and thus three "joint variables"

$$q = \{x, y, \theta\}.$$

The robot is able to translate $\{x, y\}$ (think of this as two prismatic joints) and rotate $\{\theta\}$ (think of this as a single revolute joint).

Choose a value for ρ_0 for both obstacles so that the regions of influence do not overlap. Use equations (5.5) and (5.6) to construct the repulsive potential field U_{rep} and **artificial workspace forces** F_{rep}. The math involved isn't difficult, just tedious. Be careful that you don't make silly algebra errors. You will need to find the values of $\rho(a_i(q))$ and $\nabla\rho(a_i(q))$ for each of the four vertices of the robot. Sum the repulsive forces for all vertices to get the net workspace force F.

To find the **configuration space forces *and* torque** (there should be one component of your answer for each of x, y, θ), follow Example 5.6. Each vertex a_i can be considered with coordinates a_x, a_y w.r.t. vertex a_1.

problem 5-15

You don't need to go overboard solving this problem. Just explain a method for taking random, uniformly distributed samples from the interval $[0, 1]$ and use them to construct random orientations in $SO(n)$. Do not restrict your attention to $SO(3)$.