Bayesian Region Merging Probability for Parametric Image Models[1]

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Abstract: *We describe a novel Bayesian approach to region merging, which directly uses statistical image models to determine the probability that the union of two regions is homogeneous, and does not require parameter estimation. This approach is particularly beneficial for cases in which the merging decision is most likely to be incorrect: when little information is contained in one or both of the regions and parameter estimates are unreliable. We apply the formulation to the implicit polynomial surface model for range data, and texture models for intensity images.*

1 Introduction

Uncertainty associated with an image segmentation has been a topic of recent interest in computer vision literature. Szeliski has developed uncertainty measures that apply to a given segmentation using Markov Random Field (MRF) models [5]. Chou and Brown perform Bayesian image labeling, and represent uncertainty by using general label assignments when evidence is weak, and very specific labels when the evidence is strong [1]. It is widely recognized that segmentation in general is an underconstrained problem, and that the definition of a "correct" segmentation is not possible in general [2].

At the heart of any segmentation formulation is the notion of *homogeneity* (or *uniformity*). We argue that one of the key elements to handling uncertainty in segmentation is the ability to make a probabilistic assessment about the homogeneity of a pair of regions in an image, using rigorous, statistically-based image models. To this end, we have developed the *Bayesian region merging probability*, which represents the probability that the union of two regions in an image is homogeneous, given appropriate statistical image models.

The Bayesian region merging probability is a significant contribution since:

- In the presence of uncertainty, when parameter estimates are poor, the Bayesian region merging probability gives an appropriate measure of the likelihood of merging two regions.
- The Bayesian region merging probability directly reflects the amount of uncertainty about homogeneity, which can be used by higher-level processes.
- The formalism applies to a wide class of statistical image models.
- Since the approach is Bayesian, a straightforward extension to multiple, independent image models is available.

[1] A complete version of this paper can be obtained by sending email to lavalle@cs.uiuc.edu.

2 General Formulation

With every image element, $\mathbf{x}$, we associate a random vector $\mathbf{X}$, representing the image information, which may be 3D position, intensity, color, or other information. We take some set of regions, $\mathcal{R}$, which represents a fine partition of the image into connected subsets of image elements. This could be obtained through some region-splitting procedure, or simply be an assigned grid as in [4].

For each $R_k \in \mathcal{R}$ we define the following four components, similar to those used previously in MRF contexts:

- **Parameter space:** A random vector, $\mathbf{U_k}$, which could, for instance, represent a space of polynomial surfaces.
- **Observation space:** A random vector, $\mathbf{Y_k}$, obtained as a function of the data $\mathbf{x} \in R_k$.
- **Degradation model:** A conditional density, $p(\mathbf{y_k}|\mathbf{u_k})$, which models noise and uncertainty.
- **Prior Model:** An initial parameter space density, $p(\mathbf{u_k})$.

We have shown that for two regions, R_1 and R_2, the posterior probability that $R_1 \cup R_2$ is homogeneous, given a prior probability, P_0, is determined through the following proposition:

Proposition 1 *Given the observations* $\mathbf{y_1}$ *and* $\mathbf{y_2}$, *the posterior membership probability is*

$$P(H|\mathbf{y_1},\mathbf{y_2}) = \frac{1}{1+\lambda_0\,\lambda_1(\mathbf{y_1},\mathbf{y_2})}, \qquad (1)$$

in which

$$\lambda_0 = \frac{1-P_0}{P_0} \qquad (2)$$

and $\lambda_1(\mathbf{y_1},\mathbf{y_2}) =$

$$\frac{\left[\int p(\mathbf{y_1}|\mathbf{u_1})p(\mathbf{u_1})d\mathbf{u_1}\right]\left[\int p(\mathbf{y_2}|\mathbf{u_2})p(\mathbf{u_2})d\mathbf{u_2}\right]}{\int p(\mathbf{y_1}|\mathbf{u_{12}})p(\mathbf{y_2}|\mathbf{u_{12}})p(\mathbf{u_{12}})d\mathbf{u_{12}}}. \quad (3)$$

The condition that $R_1 \cup R_2$ is homogeneous is represented by H. The λ_0 and $\lambda_1(\mathbf{y_1},\mathbf{y_2})$ ratios represent an interesting decomposition into prior and posterior factors.

3 Implicit Polynomial Surfaces for Range Data

We can apply Proposition 1 to the implicit polynomial model, pertaining to 3D surfaces. Consider a set of monomials in x_1,x_2, and x_3, in which the constants a_j, b_j, and c_j are integers that represent the exponents of each variable. An implicit polynomial equation is represented as

$$\phi(\cdot,\mathbf{u}) \equiv \sum_{j=1}^{N} u_j x_1^{a_j} x_2^{b_j} x_3^{c_j} = 0 \quad (4)$$

with $a_N = b_N = c_N = 0$. The $\cdot$ used here indicates that we have an implicit function with $\mathbf{x}$ as the variables.

It is profitable to choose some restriction of the parameter space that facilitates integrations performed in practice [3], but maintains full expressive power. We use the constraints, $\|\mathbf{u}\| = 1$ and $u_1 > 0$, to constrain the parameter space to a half-hypersphere, termed the *parameter manifold.*

The observation considered here is a function of the signed distances of the points $\mathbf{x} \in R_k$ from the surface determined by $\mathbf{u_k}$. Define $\delta(\mathbf{x},\phi(\cdot,\mathbf{u_k}))$ to be the signed distance of the point $\mathbf{x}$ to the surface described by the zero set, $\{\mathbf{x} : \phi(\mathbf{x},\mathbf{u_k}) = 0\}$. We can take the observation function as

$$y_k(R_k,\mathbf{u_k}) = \sum_{\mathbf{x}\in R_k} [\delta(\mathbf{x},\phi(\cdot,\mathbf{u_k}))]^2. \quad (5)$$

A closed-form expression for the distance of a point to a polynomial surface does not exist in general, and in practice we use a quadratic-ratio distance estimate, presented by Taubin and Cooper.

For the degradation model, we use a standard model for range-scanning error, which asserts that density, $p(\delta|\mathbf{u})$, of the signed distance of an observed point from the surface is a Gaussian random variable with zero mean and some known variance, σ^2. This model is merely a representative of possible models that can be used, and other models may be more appropriate for different imaging systems. Taking the sum of squares of Gaussian densities yields the chi-square density, $\chi^2(y_k)$, with $|R_k|$ degrees of freedom, representing the degradation model.

The prior model is defined as a uniform desnity on the parameter manifold, $p(\mathbf{u_k}) = A_N^{-1}$, in which A_N represents the area of the N dimensional parameter manifold.

4 Texture Models for Intensity Images

In this section we briefly describe how the four components of the Bayesian region merging probability model are applied to a standard class of texture models, as considered for instance in [4]. The observation space, $\mathbf{Y_k}$ is defined as the vector of intensities $X[i,j]$ in some region R_k We have an N-dimensional parameter space, $\mathbf{U_k}$ representing the texture parameters. The mean, μ_k, in R_k is represented by U_1, and U_2 represents the variance, σ_k^2. The remaining $N-2$ components are the interaction parameters, usually denoted with β. In a first-order MRF, for example, there are four parameters corresponding to interactions of $X[i,j]$ with $X[i+1,j]$, $X[i-1,j]$, $X[i,j+1]$, $X[i,j-1]$. The expressions pertain to any general order of MRF interactions, and the image element of the l^{th} parameter interaction is denoted by $T_l(x)$.

We define the prior model by assigning a uniform density to a bounded parameter space, and the degradation density for R_k can be approximated as $p(\mathbf{y_k}|\mathbf{u_k}) =$

$$\prod_{x\in R_k} \frac{1}{\sqrt{2\pi u_2}} exp\left\{\frac{-1}{2u_2}\left[x - u_1 - \sum_{l=3}^{N} u_k(T_l(x) - u_1)\right]^2\right\}. \quad (6)$$

5 Conclusions

We have developed efficient computation schemes for the evaluation of high-dimensional integrals that result from (3). We have obtained good results using the region merging probability for agglomerative clustering and for generating probability distributions of image segments and segmentations [3].

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