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Game-Theoretic Analysis of a Visibility Based Pursuit-Evasion Game in the Presence of Obstacles

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Abstract— **In this paper, we present a game theoretic analysis of a visibility based pursuit-evasion game in an environment containing obstacles. The pursuer and the evader are holonomic having bounded speeds. Both players have a complete map of the environment. Both players have omnidirectional vision and have knowledge about each other's current position as long as they are visible to each other. Under this information structure, the pursuer wants to maintain visibility of the evader for maximum possible time and the evader wants to escape the pursuer's sight as soon as possible. We present strategies for the players that are in *Nash Equilibrium*. The strategies are a function of the *value* of the game. Using these strategies, we construct a value function by integrating the *retrogressive path equations* backward in time from the termination situations provided by the corners in the environment. From these value functions we recompute the control strategies for them to obtain optimal trajectories for the players near the termination situation.**

I. Introduction

Pursuit-evasion is an interesting class of problems in the field of differential games. It involves a group of mobile agents called *pursuers* and another group of mobile agents called *evaders* involved in conflicting scenarios. Examples of such problems involve a pursuer with kinematic or dynamic constraints trying to 'catch' an evader or a pursuer trying to 'search' an evader hiding in a cluttered environment. In this work, we analyze the problem of a mobile pursuer trying to keep a mobile evader in its field-of-view in an environment containing polygonal obstacles. Both the pursuer and the evader are holonomic with bounded speeds and can see each other at the beginning of the game. The players do not have knowledge of each other's future actions. We formulate the problem of tracking as a game in which the goal of the pursuer is to keep the evader in its field-of-view for maximum possible time and the goal of the evader is to escape the pursuer's field-of-view in minimum time by breaking the line of sight around a corner.

This setting has several applications. It may be useful for a security robot to track a malicious evader that is trying to escape. The robot must maintain visibility to ensure the evader will not slip away while another party or the pursuer itself attempts to eventually trap or intercept the evader. Also, an "evader" may not be intentionally trying to slip out of view. A pursuer robot may simply be asked to continuously follow and monitor at a distance an evader performing a task not necessarily related to the target tracking game. The pursuer may somehow be supporting the evader or relaying signals to and from the evader. The pursuer may also be monitoring the evader for quality control, verifying the evader does not perform some undesired behavior, or ensuring that the evader is not in distress. Finally, the results are useful as an analysis of when escape is possible. If it is impossible to slip away, it may be desirable for the evader to immediately surrender or undertake a strategy not involving escape. In applications that involve automated processes that need to be monitored, such as in an assembly work cell, parts or sub-assemblies might need to be verified for accuracy or are determined to be in correct configurations. Visual monitoring tasks are also suitable for mobile robot applications [7]. In home care settings, a tracking robot can follow elderly people and alert caregivers of emergencies [10]. Target-tracking techniques in the presence of obstacles have been proposed for the graphic animation of digital actors, in order to select the successive viewpoints under which an actor is to be displayed as it moves in its environment [14]. In surgery, controllable cameras could keep a patient's organ or tissue under continuous observation, despite unpredictable motions of potentially obstructing people and instruments. In wildlife monitoring, deep-sea underwater autonomous vehicles (UAVs) need to navigate in cluttered environments while tracking marine species.

In this work, we use differential games to analyze a pursuit-evasion problem. The theory of deterministic pursuit-evasion was single-handedly created by R. Isaacs that culminated in his book [11]. A general framework based on the concepts in classical game theory and the notion of tenet of transition was used to analyze pursuit-evasion problems. Problems like the *Lady in the Lake*, *Lion and the Man*, *Homicidal Chauffer* and *Maritime Dogfight Problem* were introduced in this book. A modification to the classical problems involves the consideration of discrete-time versions of these problems and the application of a proper information structure to compute the value of the game [9], [8]. An exhaustive analysis of solved and partly solved zero-sum differential games is provided in [3] and [13]. Most of the classical problems in pursuit-evasion deal with players in obstacle-free space having either constraints on their motion or constraints on their control due to under-actuation. In the recent past, researchers have analyzed pursuit-evasion problems with constraints in the state space. In [16], a pursuit-evasion game is analyzed with the pursuer and the evader constrained to move on a two-dimensional conical surface in

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a three-dimensional space. A theoretical framework based on the method of characteristics is presented in [15] to address such problems. Apart from these problems, researchers have also analyzed pursuit-evasion in $\mathbb{R}^n$ [12], in non-convex domains of arbitrary dimension [1], in unbounded domains [2], and in graphs [17].

In the past, we have addressed tracking problems related to the one in this paper. In [4], we addressed the problem of a pursuer trying to track an antagonistic evader around a single corner. We partitioned the visibility region of the pursuer into regions based on the strategies used by the players to achieve their goals. Based on these partitions, we proposed a sufficient condition of escape for the evader in general environments. In [5], given the initial position of the evader in a general environment, we used the sufficient condition to compute an approximate bound on the initial positions of the pursuer from which it might track the evader. The bound depends on the ratio of the maximum speed of the evader to that of the pursuer. If the initial position of the pursuer violates this bound, the evader can escape the pursuer's sight. Moreover, we provided strategies for the evader to escape irrespective of pursuer's actions.

In this paper, we formulate the target-tracking problem as a game in which the pursuer wants to maximize the time for which it can track the evader and the evader wants to minimize it. We compute the strategies for the players that are in *Nash equilibrium* [3]. If a player follows its equilibrium strategy, it is guaranteed of a minimum outcome without any knowledge of the other player's future actions. Moreover when a pair of strategies for the players is in *Nash equilibrium* then any unilateral deviation of a player from its equilibrium strategy might lead to a lower outcome for it. Consider a situation in which the pursuer can keep the evader in sight for time t_f when the players follow their equilibrium strategies. If the evader deviates from its equilibrium strategy then the pursuer has a strategy to track it for a time greater than t_f. On the other hand, if the pursuer deviates from its equilibrium strategy then the evader can escape in time less than t_f. Hence there is no motivation for either of the players to deviate from their equilibrium strategies due to the lack of knowledge of the other player's future actions. For a pair of equilibrium strategies for the players either the evader can escape the pursuer's sight in finite time or the pursuer can track the evader forever. Hence computing them gives us the strategies sufficient for tracking or escape, whichever holds at a given point in the state space. As far as we know, this is the first work that provides the necessary and sufficient conditions for tracking and provides equilibrium strategies for the players. We use these strategies to integrate the kinematic equations of the system backward in time from the termination situations to obtain the optimal trajectories for the players.

The final results in this paper have also been presented in [6] which however uses a different modelling framework. With respect to [6], this paper differs in the following way. In [6], we model the players and the line-of-sight as a rod of variable length moving in the plane. The configuration variables of the system are the global coordinates of one end of the rod and the length and the orientation of the rod in the global frame. Then we analyze the pursuit-evasion problem as a motion planning problem in which the pursuer has to move one end of the rod in order to avoid any collision between the rod and the obstacles. In the current paper, however we use the global coordinates of the pursuer and the evader as the configuration variables to model the system. This leads to a much more elegant analysis compared to [6]. Although the fundamental principles governing the evolution of the game remain the same, the techniques used to evaluate the optimal control and then the optimal trajectories are different due to the difference in the modelling of the problem.

In Section II, we present the formulation of the game. In Section III, we present the strategies for the players that are in *Nash equilibrium*. In Section IV, we present the construction of the optimal trajectories near the termination situations around a corner. In Section V, we present conclusions and identify some future work.

II. Formulation of the Game

We consider a mobile pursuer and an evader moving in a plane with velocities $u = (u_p, \theta_p)$ and $v = (u_e, \theta_e)$ respectively. u_p and u_e are the speeds of the players that are bounded by $\overline{v}_p$ and $\overline{v}_e$ respectively. θ_p and θ_e are the direction of the velocity vectors. We use r to denote the ratio of the maximum speed of the evader to that of the pursuer $r = \frac{\overline{v}_e}{\overline{v}_p}$. They are point robots with no constraints in their motion except for bounded speeds. The workspace contains obstacles that restrict pursuer and evader motions and may occlude the pursuer's line of sight to the evader. The initial position of the pursuer and the evader is such that they are visible to each other. The visibility region of the pursuer is the set of points from which a line segment from the pursuer to that point does not intersect the obstacle region. Visibility extends uniformly in all directions and is only terminated by workspace obstacles (omnidirectional, unbounded visibility). The pursuer and the evader know each other's current position as long as they can see each other. Both players have a complete map of the environment. In this setting, we consider the following game. The pursuer wants to keep the evader in its visibility region for maximum possible time and the evader wants to break the line of sight to the pursuer as soon as possible. If at any instant, the evader breaks the line of sight to the pursuer, the game terminates. Given the initial position of the pursuer and the evader, we want to know the optimal strategies used by the players to achieve their respective goals. Optimality refers to the strategies used by the players that are in *Nash equilibrium*. Font83 The kinematic equations of the players are given as follows.

$$\dot{x}_p = u_p \cos\theta_p, \quad \dot{y}_p = u_p \sin\theta_p$$

$$\dot{x}_e = u_e \cos\theta_e, \quad \dot{y}_e = u_e \sin\theta_e$$

The above set of equations can also be expressed in the form $\dot{\mathbf{x}} = f(\mathbf{x}, u, v)$. In the next section, we present the equilibrium strategies of the players.

III. Optimal Strategies

In order to present optimal strategies, we need to define the payoff for the players in the game. Consider a play that terminates at time t_f. Since the objective of the pursuer is to increase the time of termination, its payoff function can be considered as t_f. On the other hand, since the objective of the evader is to minimize the time of termination, its payoff can be considered to be $-t_f$. Since the payoff functions of the players add to zero, this is a *zero-sum* differential game. The time of termination is a function of the initial state $\mathbf{x_0}$ and the control history during the play, $u(\cdot)$ and $v(\cdot)$.

Since the players involved in a game have conflicting goals, the concept of optimality involves the idea of *Nash equilibrium*. A set of strategies for the players is said to be in Nash equilibrium if any unilateral deviation in strategy by a player cannot lead to a better outcome for that player. Hence there is no motivation for the players to deviate from their equilibrium strategies. In case of a *zero-sum* game, the equilibrium strategies are also referred to as the *saddle-point strategies*. In scenarios where the players have no knowledge about each other's strategies, equilibrium strategies are important since they lead to a guaranteed minimum outcome for the players in spite of the other player's strategies. In this work optimal strategies refer to strategies that are in *Nash equilibrium*.

For a point $\mathbf{x}$ in the state space, $J(\mathbf{x})$ represents the outcome if the players implement their optimal strategies starting at the point $\mathbf{x}$. In this game, it is the time of termination of the game when the players implement their optimal strategies. It is also called the *value* of the game at $\mathbf{x}$. Any unilateral deviation from the optimal strategy by a player can lead to a better payoff for the other player. For example, for a game that starts at a point $\mathbf{x}$, if the evader deviates from the optimal strategy then there is a strategy for the pursuer in which its payoff is greater than $J(\mathbf{x})$ and if the pursuer deviates from the optimal strategy then there is a strategy for the evader in which its payoff is greater than $-J(\mathbf{x})$. Since this is a *zero sum* game, any strategy that leads to a higher payoff for one player will reduce the payoff for the second player.

$\nabla J = [J_{x_e} \quad J_{y_e} \quad J_{x_p} \quad J_{y_p}]^T$ denotes the gradient of the value function. The Hamiltonian of our system is given by the following expression.

$$\begin{aligned} H(\mathbf{x}, \nabla J, u, v) &= \nabla J \cdot f(\mathbf{x}, u, v) + 1 \\ &= u_e[J_{x_e}\cos\theta_e + J_{y_e}\sin\theta_e] \\ &\quad + u_p[J_{x_p}\cos\theta_p + J_{y_p}\sin\theta_p] + 1 \end{aligned}$$

Let $u^* = (u_p^*, \theta_p^*)$ and $v^* = (u_e^*, \theta_e^*)$ be the optimal controls used by the pursuer and the evader respectively. Since the pursuer is the maximizer and the evader is the minimizer, the Hamiltonian of the system satisfies the following conditions along the optimal trajectories [11]. These are called the *Isaacs* conditions.

1) $H(\mathbf{x}, \nabla J, u, v^*) \leq H(\mathbf{x}, \nabla J, u^*, v^*) \leq H(\mathbf{x}, \nabla J, u^*, v)$
2) $H(\mathbf{x}, \nabla J, u^*, v^*) = 0$

Condition 1 implies that when the players implement their optimal strategies any unilateral deviation by the pursuer leads to a smaller value for the Hamiltonian and any unilateral deviation by the evader leads to a larger value of the Hamiltonian. Moreover condition 2 implies that when the players implement their optimal controls the Hamiltonian of the system is zero. The *Isaacs* conditions are an extension of the *Pontryagin's principle* in optimization to a differential game.

Since the evader wants to minimize the time of escape, and the pursuer wants to maximize the time of escape, Isaac's first condition requires the following to be true along the optimal trajectories.

$$(u_e^*, \theta_e^*, u_p^*, \theta_p^*) = \arg\min_{u_e, \theta_e} \max_{u_p, \theta_p} H(x, \nabla J, u, v) \qquad (1)$$

We can see that the Hamiltonian is *separable* in the controls u_p and u_e i.e., it can be written in the form $u_p f_1(\mathbf{x}, \nabla J) + u_e f_2(\mathbf{x}, \nabla J)$. Hence the optimal controls for the players are given by the following expressions in terms of the gradient of the value function :

$$\begin{aligned} (\cos\theta_p^*, \sin\theta_p^*) \,||\, (J_{x_p}, J_{y_p}) \quad &\& \quad u_p^* = \overline{v}_p \\ (\cos\theta_e^*, \sin\theta_e^*) \,||\, (-J_{x_e}, -J_{y_e}) \quad &\& \quad u_e^* = \overline{v}_e \end{aligned} \qquad (2)$$

In the first and the second equation $||$ is used to denote parallel vectors. In case $J_{x_p} = 0$ and $J_{y_p} = 0$, then θ_p^* can take any value and the pursuer can follow any control strategy. Similarly if $J_{x_e} = 0$ and $J_{y_e} = 0$, then θ_e^* can take any value and the evader can follow any control strategy. These conditions represent *singularity* in the Hamiltonian.

The entire game set can be partitioned into two regions depending on the value of the game. For all the initial positions of the pursuer and the evader for which the value of the game $J(\mathbf{x})$ is finite, the evader can break the line of sight in finite time by following the strategies in equation (2). For all the initial positions of the pursuer and the evader for which the value of the game is infinite, the pursuer can track the evader forever if it follows the controls given in equation (2).

The analysis done in this section implies that if we are given the value function $J(\mathbf{x})$, then we can compute the optimal strategies for the players from equation (2). In the next section we construct such a function locally near the termination situations and present the trajectories generated by the players when they follow the optimal strategies near the termination situations.

IV. Construction of Optimal Trajectories

In this section, we present the trajectories generated by the optimal control laws near termination situations. The *game set* is the set of all states in $\mathbb{R}^4$ such that the players are in the free workspace and can see each other. Hence the boundary of the game set is the same as the boundary of the configuration space obstacles. The boundary of the game set consists of two kinds of configurations of the pursuer and the evader. Refer to Figure IV. The first kind of boundary points consists of states in which either the pursuer or the evader or both lie on the boundary of the workspace. At no point in time, the state of the game can cross the boundary at such a point as this is equivalent to either of the players penetrating into an obstacle in the workspace. The second kind of boundary points consists of states in which the line of sight between the pursuer and the evader grazes the boundary of an obstacle and this set of points on the boundary of the game set is called the *Target set* [11]. At any point in time, if the current state of the game lies on the target set, then it can cross the boundary according to the rules of the game since in the workspace this is equivalent to breaking the mutual visibility between the players which results in the termination of the game. Since we are interested in situations where the mutual visibility between the players can be broken, we are only interested in the part of the boundary that forms the target set.

In this game, termination occurs only when the evader can break the line of sight to the pursuer around a corner. Every corner in the environment presents an opportunity for the evader to break the line of sight. Hence every corner presents a termination situation for the game.

If the state of the system lies on the target set then a vertex of some obstacle is incident on the mutual visibility line between the pursuer and the evader. The evader cannot guarantee termination at every point on the target set. Figure 1 shows a configuration in which the state of the system lies on the target set. Let l_p denote the distance of the vertex from the pursuer. Let l denote the distance between the pursuer and the evader. The evader can force termination if and only if the maximum angular velocity of the evader around the corner is greater than the maximum angular velocity achievable by the pursuer around the corner. This can happen if and only if the following condition holds :

$$\frac{l_p}{l} > \frac{1}{1+r} \tag{3}$$

Hence we can further subdivide the target set, depending on whether the evader can guarantee termination at that point. The part of the target set where evader can guarantee termination regardless of the choice of the controls of the pursuer is called the *usable part* (UP). Given any initial position of the pursuer and the evader, the game will always terminate on the UP.

Now we present the equations characterizing the target set around a vertex of an obstacle; see Figure 1. The figure shows a configuration of the bar in which a vertex, v, that lies on the line of sight between the pursuer and the evader. Hence the current state of the bar lies on the target set. We want the equation of the hyperplane that characterizes the target set generated by v. Let (x_p, y_p, x_e, y_e) be the state of the system on the target set and (x^o, y^o) be the coordinates of the vertex of the obstacle. We can write the following equation of constraint :

$$\frac{y^o - y_e}{x^o - x_e} = \frac{y^o - y_p}{x^o - x_p}$$

Hence the target set is characterized by the following equation :

$$F(x_p, y_p, x_e, y_e) = (y^o - y_p)(x^o - x_e) - (y^o - y_e)(x^o - x_p) = 0 \tag{4}$$

Since the above equation applies to any point on the target set, equation (3) also characterizes the UP of the target set. Since the target set is on the boundary of the game set, it is 3-dimensional and hence can be represented by 3 independent variables. Let the independent variables representing the target set be chosen as the following:

$$s_1 = x_e - x^o, \quad s_2 = y_e - y^o, \quad s_3 = x_p - x^o$$
$$\implies y_p = y^o + \frac{s_2 s_3}{s_1}$$

The value function at every point on the UP is 0. Hence the partial derivative of the value function along s_1, s_2 and s_3 is zero. Let the components of ∇J on the UP be denoted as $[J^0_{x_p} \quad J^0_{y_p} \quad J^0_{x_e} \quad J^0_{y_e}]^T$.

$$J^0_{s_1} = 0 = J^0_{x_e} - J^0_{y_p}\frac{s_2 s_3}{s_1^2}, \quad J^0_{s_2} = 0 = J^0_{y_e} + J^0_{y_p}\frac{s_3}{s_1}$$
$$J^0_{s_3} = 0 = J^0_{x_p} + J^0_{y_p}\frac{s_2}{s_1} \tag{5}$$

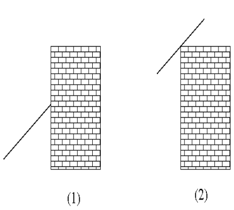
From the Isaacs second condition, the following equation holds at every point in time :

$$-\overline{v}_e\sqrt{J^2_{x_e} + J^2_{y_e}} + \overline{v}_p\sqrt{J^2_{x_p} + J^2_{y_p}} + 1 = 0 \tag{6}$$

Substituting Equations (5), (6) and (7) into Equation (8), we get the following expression for $J^0_{y_p}$:

$$\mid J^0_{y_p} \mid = \frac{1}{(\sqrt{\frac{s_2^2}{s_1^2} + 1})(\overline{v}_e\sqrt{\frac{s_3^2}{s_1^2}} - \overline{v}_p)} \tag{7}$$

From equation (3), we can conclude that on the UP, $\mid \frac{s_3}{s_1} \mid > \frac{\overline{v}_p}{\overline{v}_e}$ and hence the R.H.S. of the above equation is always positive. Hence $J^0_{y_p}$ can have two possible values differing just by a sign. In the termination condition shown in Figure 1, $J^0_{y_p}$ is positive since the value of the game increases when we perturb the pursuer position vertically upwards.



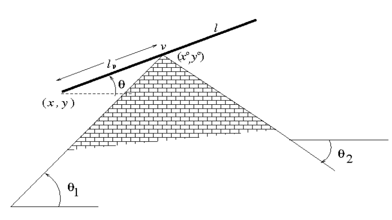


Fig. 1. State of the system on the target set.

Depending on the position of the corner and the orientation of the pursuer and the evader at the termination situation, we can eliminate one of the possible values of $J^0_{y_p}$.

Now we use the following theorem to obtain the value function along the optimal trajectories backwards in time.

Theorem[11]: Along the optimal trajectory, the following equation holds.

$$\frac{d}{dt}\nabla J[\mathbf{x}(t)] = -\frac{\partial}{\partial x}H(\mathbf{x}, \nabla J, \mathbf{u}^*, \mathbf{v}^*)$$

The above equation is called the *retrogressive path equation* (RPE). The RPE is a differential equation for the $\nabla J(\mathbf{x})$ along the optimal trajectories in terms of the optimal controls. Substituting the optimal controls of the players as a function of $\nabla J(\mathbf{x})$ from Equation (2) into the RPE leads to a set of ordinary differential equations for $\nabla J(\mathbf{x})$. For our system, the RPE gives the following set of differential equations:

$$\begin{aligned}\mathring{J}_{x_p} = 0 \Longrightarrow J_{x_p} = J^0_{x_p}, \quad \mathring{J}_{y_p} = 0 \Longrightarrow J_{y_p} = J^0_{y_p} \\ \mathring{J}_{x_e} = 0 \Longrightarrow J_{x_e} = J^0_{x_e}, \quad \mathring{J}_{y_e} = 0 \Longrightarrow J_{y_e} = J^0_{y_e}\end{aligned} \qquad (8)$$

Substituting $\nabla J(\mathbf{x})$ into the optimal controls in equation (2) yields the control strategies for the players :

$$\begin{aligned}(\cos\theta^*_p, \sin\theta^*_p) \,||\, (J^0_{x_p}, J^0_{y_p}) \quad \& \quad u^*_p = \overline{v}_p \\ (\cos\theta^*_e, \sin\theta^*_e) \,||\, (-J^0_{x_e}, -J^0_{y_e}) \quad \& \quad u^*_e = \overline{v}_e\end{aligned} \qquad (9)$$

Substituting the control laws for the players into the kinematic equation leads to the optimal trajectories in retro time. Let $(x^f_p, y^f_p, x^f_e, y^f_e)$ be the state of the system at the termination situation on the UP. From equation (9), the value of $J^0_{y_p} = \pm c_1 cos\theta_f$, where $c_1 = \frac{1}{\overline{v}_e|\frac{x^o - x^f_e}{x^o - x^f_p}| - \overline{v}_p}$ and $\tan\theta_f = \frac{y^f_e - y^o}{x^f_e - x^o}$. The optimal trajectory of the pursuer as a function of retro-time is given by

$$x_p(\tau) = x^f_p \pm \tau\overline{v}_p \sin\theta_f, \quad y_p(\tau) = y^f_p \mp \tau\overline{v}_p \cos\theta_f \quad (10)$$

The optimal trajectory of the evader as a function of retro-time is given by

$$x_e(\tau) = x^f_e \mp \tau\overline{v}_e \sin\theta_f, \quad y_e(\tau) = y^f_e \pm \tau\overline{v}_e \cos\theta_f \quad (11)$$

Since ∇J is constant along an optimal trajectory, from the expression of the optimal strategies of the players, we see that they are straight lines. Moreover from equations (14) and (15), we conclude that the players move parallel to each other in opposite directions, perpendicular to the line of sight at the termination situation. Given a termination situation, this leads to two kinds of trajectories for the players as shown in Figure 2. Now we show that only one of these two kinds can lead to termination.

Referring to Figure 3, let **p** and **e** be positions of the pursuer and the evader at a termination situation. Consider a small amount of perturbation in the pursuer's position in the positive y-direction. Let the new position of the pursuer be $\mathbf{p}'$. The value of the game at $(x_{p'}, y_{p'}, x_e, y_e)$ is greater than zero since the evader cannot terminate the game instantly. Hence J_{y_p} is greater than zero at (x_p, y_p, x_e, y_e). The velocity of the pursuer is perpendicular to the line-of-sight between the pursuer and the evader at the termination situation. Since $J_{y_p} > 0 \Longrightarrow \sin\theta^*_p > 0 \Longrightarrow 0 < \theta^*_p < \pi$ at the termination situation. Hence the pursuer approaches the termination situation in the direction shown in the figure. Since the velocity of the evader is in the opposite direction, the evader approaches the termination situation in the direction shown in the figure. Repeating the above analysis for all orientation of the termination configuration and the obstacle leads to the conclusion that at the termination situation the evader moves towards the obstacle and the pursuer moves away from the obstacle. This leads to a unique set of optimal trajectories from every point on the UP.

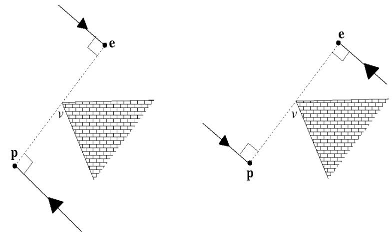


Fig. 2. Optimal trajectories to a termination situation

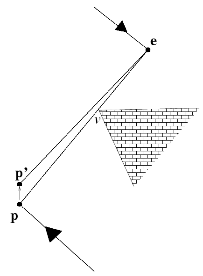
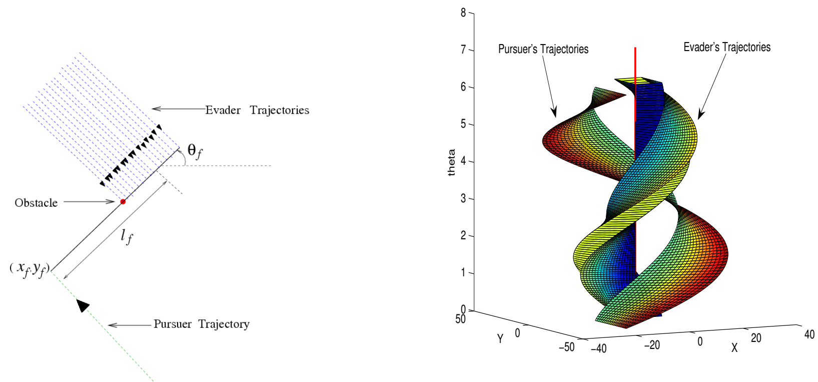


Fig. 3. A configuration of the bar on the target set.

For a general environment in the plane, the optimal trajectories lie in $\mathbb{R}^4$. In order to depict them in $\mathbb{R}^3$, we need to consider a subspace of the optimal paths terminating at a corner. In the following example, for each corner in the environment we show the subspace of the optimal paths that have a fixed distance of the pursuer from the corner at the termination situation. The value of the speed ratio, r, is 0.66. Figure 4 shows the optimal trajectories for the players in a simple environment containing a point obstacle at the origin. The line of sight between the pursuer and the evader is broken if it passes through the origin. The evader wants to minimize the time required to break the line of sight and the pursuer wants to maximize it. Let $(x^f_p, y^f_p, x^f_e, y^f_e)$ represent the orientation of the bar at the termination situation. Figure 4(a) shows the optimal trajectories of the players for a constant value of $(x^f_p,\ y^f_p)$. Figure 4(b) shows the optimal trajectories for every orientation of the line-of-sight between the pursuer and the evader at the termination situation. The z axis represents the angle that the line-of-sight makes with the

Fig. 4. Optimal trajectories for an environment having a single point obstacle



(a) Optimal Trajectories in the plane (b) Optimal Trajectories across a section in $\mathbb{R}^4$

horizontal axis at the termination situation. A cross-section parallel to the xy-plane gives the optimal trajectories of the players in a plane for a given θ_f. The line in the middle denotes the point obstacle. The inner spiral is formed by the optimal trajectories of the evader and the outer spiral is formed by the optimal trajectory of the pursuer. For any point on the spiral, the value of the game is directly proportional to its radial distance from the point obstacle.

Finally, we present below conclusions and identify some work for the future.

V. Conclusion and Future work

In this paper, we have addressed a visibility based pursuit-evasion game in an environment containing obstacles. The pursuer and the evader are holonomic having bounded speeds. The pursuer wants to maintain visibility of the evader for maximum possible time and the evader wants to escape the pursuer's sight as soon as possible. Both players have knowledge about each other's current position. Under this information structure, we have presented necessary and sufficient conditions for surveillance and escape. We have presented strategies for the players, which are in Nash Equilibrium. The strategies are functions of the value of the game. Using these strategies, we have constructed a value function by backward integration of the adjoint equations from the termination situations provided by the corners in the environment. From the value functions we have recomputed the control strategies for the players to obtain optimal trajectories for the players near the termination situation. We have shown that the optimal strategies for the players dictate moves on straight lines parallel to each other in opposite directions towards a termination situation. We have shown a subspace of the optimal trajectories for a point obstacle, a corner and a hexagonal obstacle in space.

Among the future work are developing complete solutions for a polygonal environment containing multiple obstacles and the extension of the techniques developed here to address the problem of multiple pursuers trying to attack an evader. For the former, construction of various types of singular surfaces will be needed.

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